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Spectral Properties of Some Operator, Rationally Depending on Parameter

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Abstract: It is investigated the spectral properties of some operator, rationally depending on parameter in Hilbert space. It is known that the problem of oscillations of viscous liquid located in a stationary vessel and having a free surface, leads to the investigation of such type operator equations. Methods of investigations in this work are the methods of functional analysis, spectral theory of operators, theory of functions and multiparameter system of operators. It is famous that the notions of completeness of eigen and associated (e.a.) vectors, multiple completeness of them, question of existence of bases of e.a. vectors, multiple bases, asymptotic of eigenvalues are the fundamental directions in spectral theory of operators in Hilbert space.
In this paper the authors are proved the possibility of existence of two bases of e.a. vectors of considerable equation in the Hilbert space, proved the existence of two sequences of eigenvalues of operator equation and the asymptotic of corresponding eigenvalues.
Keywords: Vessel; Space; Eigen; Equation; Vector.

1. Introduction

Krein [1] had shown that the problem of oscillations of viscous liquid located in a stationary vessel and having a free surface, leaded to the equation

$$y = \lambda G y + \frac{1}{\lambda} H y \qquad (1)$$

Later, Allakhverdiev proved the existence of two kind of multiple completeness system of e.a.vectors of the operator pencil, rationally depending on parameter.

The problems of existence of two kind of the bases of e.a. vectors of the operator $\mu G + \frac{1}{\mu} H$ when $G,\ H \in \sigma_\infty \ \ G > 0,\ \ H \ge 0$, λ $-$complex numbers were considered in the [2].

Further, it is received a number of significant additions to the work of Askerov, *et al.* [2].

For example, if $4(Gx,x)(Hx,x) \prec (x,x)^2, x \in H, x \ne 0$ and $G.\ H \in \sigma_\infty, G > 0$ then e.a. vectors of the equation (1) form two system of the Riess bases , composed, respectively, of the eigenvectors of the first and second kind, and that the appropriate complete sequence of characteristic values of a first kind $\lambda_1^{(1)} \le \lambda_2^{(1)} \le \lambda_3^{(1)} \le \ldots$ of the equation $E - \lambda G$ with the asymptotic

$$\lambda_n^{(1)} = \mu_n^{-1}(G)(1 + o(1)) \qquad (2)$$

and if $H > 0$, then also for relevant sequence of characteristic numbers of the second kind

$$\lambda_n^{(2)} = \mu_n(H)(1 + o(1)) \qquad (3)$$

So, asymptotic of characteristic numbers of the first kind of equation (1) is determined by characteristic numbers of the operator G

And the asymptotic of the characteristic numbers of the second kind of (1) are determined by the eigenvalues of operator H .

If these operators have finite order, then there is a two complete completeness of the system of its eigen and adjoint vectors in Hilbert space.

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2. The Operator, Rationally Depending on Parameter

We consider the equation

$$y = Ty + \lambda Gy + \frac{1}{\lambda} Hy \tag{4}$$

Laptev gave the original manner when investigation of the equation (4) leads to the investigation of equation

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} T & H-G \\ H-G & T \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + \lambda \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} \tag{5}$$

in the direct sum $\tilde{\mathrm{H}}$ of two copies of the space H .It is known $\tilde{\mathrm{H}}$ is the Hilbert space. When the complete continuous operators $G,\ H, T \in \sigma_\infty$ satisfied to conditions $G, H \in \sigma_p, T \in \sigma_\infty$ then the assertion of Keldysh theorem is true. Hence the e.a. vectors of (5) form the complete system in the space $\tilde{\mathrm{H}}$. Last means the two multiple completeness of the system of e. a.vectors (4) in the space H .

Theorem1. Let the following conditions are true:

$$T, G \in \sigma_p, p<1,\ \ HG^{-p} \in \sigma_\infty, GH^{-p} \in \sigma_\infty, TG^{-p} \in \sigma_\infty, KerG = KerH = \{\vartheta\}$$

Then there are two multiple bases with brackets of e.a. vectors of equation (4). Proof of Theorem1.
Theorems about multiple expansions on the root subspaces of the operator pencil

$$L(\lambda) = E - A_0 - \lambda A_1 B - \ldots - \lambda^{n-1} A_{n-1} B^{n-1} - \lambda^n B^n \tag{6}$$

are proved in the works of RM Dzhabarzadeh , V.N. Vizitey, A.S. Marcus with the proviso that operators $A_i B^{-i}$ are bounded if $\varliminf_{k\to\infty} k\mu_k^{-p} = \infty,$ and operators $A_i B^{-i}$ are completely continuous if $\varliminf_{k\to\infty} k\mu_k^{-p} < \infty.$

Through μ_k are designated the sequence, in order of increasing various modules of the characteristic values of the operator B , taking into account of their multiplicities.
The proof of the Theorem 1 is carried out similarly to that of above-stated theorems.
We investigate the equation (5) so there is the closed connection between the e.a. vectors of the equation (4) and (5).

If $KerG = KerH = \{\vartheta\}$ then the operator $\begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix}$ has inverse $\begin{pmatrix} G^{-1} & 0 \\ 0 & H^{-1} \end{pmatrix}$

The operator $\begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix}$ is complete continuous and have the finite order equally to p.

From the condition of Theorem 1 follows that operator

$$\begin{pmatrix} T & H-G \\ H-G & T \end{pmatrix}\begin{pmatrix} G^{-p} & 0 \\ 0 & H^{-p} \end{pmatrix} = \begin{pmatrix} TG^{-p} & H^{1-p} - GH^{-p} \\ HG^{-p} - G^{1-p} & TH^{-p} \end{pmatrix}$$

is bounded in the space $\tilde{\mathrm{H}}$, then there is basis with brackets from their e.a. vectors of the equation (5) in the space $\tilde{\mathrm{H}}$. Last means there is two multiple basis with brackets on e.a. vectors of equation (4).

Theorem 2. Let the conditions of the Theorem1 are true then equation (5) has two corresponding sequences of eigen values and characteristic numbers with the asymptotic (6) and (7), correspondingly.

Proof of the Theorem 2. Theorem 1 stated the equation $x = Tx + \lambda Gx + \frac{1}{\lambda} Hx$ has two Riesz bases, composed, respectively, of e.a. vectors of the first and second kind. We prove the asymptotic $\lambda_n^{(1)} = \mu_n^{-1}(G)(1 + o(1))$ of complete sequence $\lambda_1^{(1)} \le \lambda_2^{(1)} \le \lambda_3^{(1)} \le \ldots$ of characteristic values of a first-order of the operator (5) is defined by the characteristic numbers of the operator G .Also for relevant sequences characteristic numbers of the second kind $\lambda_n^{(2)} = \mu_n(H)(1+o(1))$ the asymptotic $\lambda_n^{(2)} = \mu_n(H)(1+o(1))$ is defined by the eigenvalues of the operator H.

Let be $G \in \sigma_p, p < 1, G > 0,\ KerG = \{\vartheta\}$. We denote characteristic numbers of operator G in order of increasing through λ_k .Then there is the subset of closed contours $\Gamma_k,\ k \to \infty$ with radii $r_k \to \infty$ such, that for all $\lambda \in \Gamma_k,\ k \to \infty$

$$\frac{\lambda_k^{1-p}}{\Delta\lambda_k} \to 0,\ \Delta\lambda_k = \lambda_{k+1} - \lambda_k \to \infty,\ r_i = \lambda_k + \frac{\Delta\lambda_k}{2},\ k \to \infty.$$

Really, from the expressions

$$(E - T - \lambda G - \frac{1}{\lambda}H)^{-1} = (E - \lambda G)^{-1}(E - T(E - \lambda G)^{-1} - \frac{1}{\lambda}H(E - \lambda G)^{-1})^{-1} =$$

$$= (E - \lambda G)^{-1}(E - TG^{-p}(G^{-1} - \lambda E)^{-1}G^{p-1} - \frac{1}{\lambda}HG^{-p}(G^{-1} - \lambda E)^{-1}G^{p-1})^{-1} \tag{7}$$

Under the conditions of the Theorem2 for sufficiently large values k operators

$TG^{-p}(E - \lambda G)^{-1}G^{p-1} - \frac{1}{\lambda}HG^{-p}(E - \lambda G)^{-1}G^{p=1}$ and

$H^{p-1}(H^{-1} - \frac{1}{\lambda}E)^{-1}H^{-p}T - \lambda H^{p-1}(H^{-1} - \frac{1}{\lambda}E)^{-1}H^{-p}$ have the norms less than 1 .Then for all $\lambda \in \Gamma_k,\ k > N$ operators

$E - TG^{-p}(E - \lambda G)^{-1}G^{p-1} - \frac{1}{\lambda}HG^{-p}(E - \lambda G)^{-1}G^{p=1}$ and

$E - H^{p-1}(H^{-1} - \frac{1}{\lambda}E)^{-1}H^{-p}T - \lambda H^{p-1}(H^{-1} - \frac{1}{\lambda}E)^{-1}H^{-p}$ have steadily bounded inverses/

So asymptotic behavior $\lambda_n^{(1)} = \mu_n^{-1}(G)(1+o(1))$ of the characteristic number of the first kind of the operator equation (5) is defined by the characteristic numbers of operator G and the asymptotic $\lambda_n^{(2)} = \mu_n(H)(1+o(1))$ of the characteristic numbers of the equation(5) of the second kind is defined by the eigenvalues of the operator H

3. New Approach to the Study of the Problem (4)

When operators T, G, H are bounded in Hilbert space . We introduce the symbols for consideration

$$\lambda = \lambda_1, \frac{1}{\lambda} = \lambda_2$$

Then we obtain an equation $x = Tx + \lambda_1 Gx + \lambda_2 Hx,\ \ x \in \mathrm{H}$ with two unknown parameters. Supplement this equation with equation

$t_0 y = \lambda_1 t_1 y + \lambda_2 t_2 y,\quad y \in R_2$ where operators,

$$t_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, t_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

All entered in (8) operators are selfadjoint and act in the space R_2.
Thus, we come to the two-parameter system

$$\begin{aligned} x &= Tx + \lambda_1 Gx + \lambda_2 Hx \\ t_0 y &= \lambda_1 t_1 y + \lambda_2 t_2 y \end{aligned} \tag{9}$$

We need some definitions for the particularly case when the number of parameters in multiparameter system is equal to 2.

Definition 1. [3-5] $\lambda = (\lambda_1, \lambda_2) \in C^2$ is an eigenvalue of the system (9) if there are non-zero elements $x \in \mathrm{H},\ y \in R_2$ such that (9) is satisfied, and decomposable tensor $x \otimes y \in \mathrm{H} \otimes R_2$ is called the eigenvector of (9) corresponding to eigenvalue $\lambda = (\lambda_1, \lambda_2) \in C^2$

Definition 2. [3-5] The operators T^+, G^+, H^+, t_0^+, t_1^+, t_2^+ are induced into the space $\mathrm{H}\otimes R_2$ by operators T, G, H, t_0, t_1, t_2 correspondingly, if on each decomposable tensor $x\otimes y$ of tensor product space $\mathrm{H}\otimes R_2$ we have $T^+ = T\otimes E_2$, $G^+ = G\otimes E_2$, $H^+ = H\otimes E_2$, $t_0^+ = t_0\otimes E_2$, $t_1^+ = t_1\otimes E_2$, $t_2^+ = t_2\otimes E_2$, on all other elements of $\mathrm{H}\otimes R_2$ the operators $T^+, G^+, H^+, t_0^+, t_1^+, t_2^+$ are defined on linearity and continuity. E_1, E_2 are the identity operators in the spaces H and R_2., correspondingly.

Definition3. A tensor z_{m_1,m_2} is named $(m_1,m_2)-$th the associated vector to an eigenvector of the system (9) if the following conditions are satisfied

$$T^+ z_{k_1,k_2} = G^+ z_{k_1-1,k_2} + H^+ z_{k_1,k_2-1},$$
$$(t_0^+) z_{k_1,k_2} = t_1^+ z_{k_1-1,k_2} + t_2^+ z_{k_1,k_2-1} \qquad k_s \le m_s; i=1,2;\ s=1,2.$$

(k_1,k_2) - arrangements from set of the whole nonnegative numbers on 2 with possible recurring and zero.

Under canonical system of e.a.vectors (this definition is generalization of the definition of canonical system in the case of polynomial pencils in one parameter) for (λ_1,λ_2) we understand

The set of vectors $\left\{z_{i_1,i_2}^{(k)}\right\}_{0\le i_r\le m_r}$, $r=1,2$ possessing the following properties: elements $z_{0,0}^{(k)}$ form basis of a eigen subspace $M(\lambda^0)$; there is $z_{0,0}^{(1)}$ eigenvector which multiplicity reaches a possible maxima p_1+1; there is $z_{0,0}^{(k)}$ eigenvector which is not expressing linearly through $z_{0,0}^{(1)},\ldots, z_{0,.0}^{(k-1)}$ which sum of multiplicities reaches a possible maxima p_k+1. Let's designate through $M(\lambda_1,\lambda_2)$ a subspace tense on e.a. vectors of the system (9), corresponding to an eigenvalue (λ_1,λ_2).

Linearly-independent elements form a chain $\left(z_{i_1,i_2}\right)\subset \mathrm{H}\otimes R_2$ of a set of e.a vectors of (9). The multiplicity of eigenvalue designates the greatest number of associated vectors to $z_{0,0}$ a plus 1.

The sum $p_1+p_2+\ldots+p_s+s$ is a multiplicity of an eigenvalue (λ_1,λ_2).

Elements $\left\{z_{i_1,i_2}^{(k)}\right\}_{0\le i_r\le m_r}$, $r=1,2,$ … form a chain of e.a. vectors for every fixed value k, $k=1,2,\ldots,s$.

Definition4. From [3-5] for the system (9) are introduced the analogues of the Cramer's determinants, when the number of equations is equal to the number of variables, by the rules : on decomposable tensor $x\otimes y$ operators Δ_i are defined with help of the matrices

$$\Delta_0 = \begin{pmatrix} G^+ & H^+ \\ t_1^+ & t_2^+ \end{pmatrix} = \begin{pmatrix} -H^+ & 0 \\ 0 & G^+ \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} (E_1-T)^+ & H^+ \\ t_0^+ & t_2^+ \end{pmatrix} = \begin{pmatrix} 0 & -H^+ \\ -H^+ & (E_1-T)^+ \end{pmatrix} \qquad (10)$$

$$\Delta_2 = \begin{pmatrix} G & (E_1-T)^+ \\ t_1^+ & t_0^+ \end{pmatrix} = \begin{pmatrix} -(E_1-T)^+ & G^+ \\ G^+ & 0 \end{pmatrix}$$

In multiparameter theory are introduced the operators

$$\Gamma_1 = \Delta_0^{-1}\Delta_1 = \begin{pmatrix} -(H^+)^{-1} & 0 \\ 0 & (G^+)^{-1} \end{pmatrix}\begin{pmatrix} 0 & -H^+ \\ -H^+ & (E_1-T)^+ \end{pmatrix} = \begin{pmatrix} 0 & E \\ -G^+H^= & (G^+)^{-1}(E_1-T)^+ \end{pmatrix}\text{-}$$

$$\Gamma_2 = \Delta_0^{-1}\Delta_2 = \begin{pmatrix} -(H^+)^{-1} & 0 \\ 0 & (G^+)^{-1} \end{pmatrix} \begin{pmatrix} -(E \ -T)^+ & G^+ \\ G^+ & 0 \end{pmatrix} = \begin{pmatrix} (H^+)^{-1}(E_1 - T)^+ & -(H^+)^{-1}G^+ \\ E & 0 \end{pmatrix}$$

After transposition we obtain

$$\Gamma_1 = \Delta_0^{-1}\Delta_1 = \begin{pmatrix} 0 & E \\ -GH \otimes E_2 & G^-(E_1 - T) \otimes E_2 \end{pmatrix}$$

$$\Gamma_2 = \begin{pmatrix} H^{-1}(E_1 - T) \otimes E_2 & -H^{-1}G \otimes E_2 \\ E & 0 \end{pmatrix} \qquad (11)$$

From [6] under the conditions of the Theorem2 we have that the system of e.a.vectors of the system (9) coincides with the system of e.a. of each operator $\Gamma_1 = \Delta_0^{-1}\Delta_1$ and $\Gamma_2 = \Delta_0^{-1}\Delta_2$.

If the vectors of (9) form the complete system in the space $H \otimes R_2$ then the first components of them being the e.a. vectors of (5) form the complete system in the space H .

4. Conclusion

It is proved the existence of two multiple bases of e.a.vectors, asymptotic of characteristic numbers, and sufficient and necessary conditions of completeness of e.a. vectors of the equation (5) in the Hilbert space.

References

[1] Krein, S. Q., 1964. "Probuid ems of oscillationsof viscous loquid located in a stationary vessel and having a free surface." *DAN*, vol. 152.

[2] Askerov, N. Q., Krein, S. Q., and Laptev, Q. I., 1964. "On one class of nonselfadjoint boundary problem." *DAN*, vol. 155, pp. 499-502.

[3] Atkinson, F. V., 1968. "Multiparameter spectral theory." *Bull. Amer. Math. Soc.*, vol. 74, pp. 1-27.

[4] Browne, P. J., 1974. "Multiparameter spectral theory." *Indiana Univ. Math. J.*, vol. 24, p. 3.

[5] Sleeman, B. D., 1978. *Multiparameter spectral theory in Hilbert- space*. London: Pitnam Press. p. 118.

[6] Dzhabarzadeh, R. M., 1999. "Spectral theory of multiparameter system of operators in Hilbert space." *Transactions of NAS of Azerbaijan*, vol. 1-2, pp. 33-40.