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Spectra of Graphs with End Vertices Appended to All Vertices of the Base Graph: The Golden Ratio and Energy

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Abstract: In this paper, we determine the spectra of graphs obtained by appending h end vertex to all vertices of a defined class of graphs called the *base graph*. The end vertices allow for a quick solution to the eigen-vector equations satisfying the characteristic equation, and the solutions to the eigenvalues of the base graph arise. We determine the relationship between the eigenvalues of the base graph and the eigenvalues of the new graph as constructed above, and determine that if α is an eigenvalue of the base graph, then $\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + 4h}}{2}$ is an eigenvalue of the constructed graph. There has been much interest in spectra of graphs with the *golden ratio property* (graphs having eigenvalues $\lambda = \pm 1.618 = \pm\frac{1+\sqrt{5}}{2}$ or $\lambda = \pm 0.618 = \pm\frac{-1+\sqrt{5}}{2}$ of the adjacency matrix). We then determine that if $\alpha = +1$ is an eigenvalue of the base graph, then $\lambda = \frac{1+\sqrt{5}}{2}$ is an eigenvalue of the graph constructed by adding one end vertex to each of the vertices in the base graph. Similarly, if $\alpha = -1$ is an eigenvalue of the base graph, then $\lambda = \frac{-1+\sqrt{5}}{2}$ is an eigenvalue of the graph constructed by adding one end vertex to each of the vertices in the base graph.

Finally, we determine the spectra for such graphs where the base graph is one of the well-known classes of graphs, namely the complete; complete split-bipartite, cycle, path, wheel and star graphs. There has been much interest in regular graphs with four distinct eigenvalues of the adjacency matrix – see van Dam [2]. In this paper we also determine the spectra of a non-regular class of graphs with diameter 3, which has 4 distinct eigenvalues; namely the complete sun graph which consists of the complete graph with end vertices appended to each of its vertices.

The energy of a graph, defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph, is related to the sum of π-electron energy in a molecule, represented by a molecular graph, where the vertices represent the atoms and the edges represent the bonds between the atoms. See [3]. We also determine the energy of the graph, constructed from the base, graphs for each of the above classes of graphs.

Keywords: Spectra of graphs; Golden ratio; Graphs with small number of distinct eigenvalues; Graphs with many end vertices; Energy of graphs.

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1. Introduction

There has been much interest in spectra of graphs with the *golden ratio property* (having eigenvalues $\lambda = \pm 1.618 = \pm \frac{\sqrt{5}+1}{2}$ or $\lambda = \pm 0.618 = \pm \frac{\sqrt{5}-1}{2}$ see [1]) and in regular graphs with four distinct eigenvalues of the adjacency matrix (see [2]). In this paper we determine the spectra of a non-regular class of graphs with diameter 3, by appending end vertices to each of the vertices of a base graph. We then determine that whenever one vertex is appended to certain graphs, the result is a graph with the Golden ratio property.

Generally, we determine the spectra of graphs obtained by appending h end vertex to all vertices of a defined class of graphs called the *base graph*. The end vertices allow for a quick solution to the eigen-vector equations satisfying the characteristic equation, and the solutions to the eigenvalues of the base graph arise. We determine the relationship between the eigenvalues of the base graph and the eigenvalues of the new graph as constructed above, and determine that if α is an eigenvalue of the base graph, then $\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + 4h}}{2}$ is an eigenvalue of the constructed graph.

We then determine that if $\alpha = +1$ is an eigenvalue of the base graph, then $\lambda = \frac{1+\sqrt{5}}{2}$ is an eigenvalue of the graph constructed by adding one end vertex to each of the vertices in the base graph. Similarly, if $\alpha = -1$ is an eigenvalue of the base graph, then $\lambda = \frac{-1+\sqrt{5}}{2}$ is an eigenvalue of the graph constructed by adding one end vertex to each of the vertices in the base graph.

Finally, we determine the spectra for such graphs where the base graph is one of the well-known classes of graphs, namely the complete, complete split-bipartite, cycle, path, wheel and star graphs. We also determine the energy of the constructed graph for each of these classes of graphs.

2. End Vertices Appended To Each Vertex in a Graph

Let the generalized sun graph $GSun(h, p)$ be a graph which consists of the base graph G on p vertices, with h end vertices appended to each of the p vertices in the graph G. Then the graph $Gsun(h, p)$ has $n = p(h+1)$ vertices, and the (nxn) adjacency matrix of $GSun(h, p)$ is:

$$A(GSun(h,p)) = \begin{bmatrix} A(G) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

Theorem 2.1

If α_j are the eigenvalues of $A(G)$, $1 \le j \le p$, then $\{\lambda_{2j-1}, \lambda_{2j}\} = \frac{\alpha_j \pm \sqrt{\alpha_j^{\ 2} + 4h}}{2}$ are two eigenvalues of $A(GSun(h,p))$, $1 \le j \le p$. The remaining $p(h-1)$ eigenvalues of $A(GSun(h,p))$ are $\lambda = 0$.

Proof

Let $\underline{x} = (x_1, x_2, x_3, \ldots, x_n)$ be an eigenvector of $A(GSun(h,p))$, with eigenvalue λ. Then we have:

$$A(GSun(h,p))\underline{x} = \lambda \underline{x}$$

$$\Rightarrow \begin{bmatrix} A(G) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix} \underline{x} = \lambda \underline{x}$$

$$\Rightarrow \begin{bmatrix} A(G)_1 (x_1,\dots,x_p)^T + \sum_{i=1}^{h} x_{ip+1} \\ A(G)_2 (x_1,\dots,x_p)^T + \sum_{i=1}^{h} x_{ip+2} \\ \vdots \\ A(G)_k (x_1,\dots,x_p)^T + \sum_{i=1}^{h} x_{ip+k} \\ \vdots \\ A(G)_p (x_1,\dots,x_p)^T + \sum_{i=1}^{h} x_{ip+p} \\ x_1 \\ x_{21} \\ \vdots \\ x_p \\ \vdots \\ x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_k \\ \vdots \\ \lambda x_p \\ \lambda x_{p+1} \\ \lambda x_{p+2} \\ \vdots \\ \lambda x_{2p} \\ \vdots \\ \lambda x_{hp+2} \\ \lambda x_{hp+2} \\ \vdots \\ \lambda x_{hp+p} \end{bmatrix} \quad labelled \quad \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \\ \vdots \\ p \\ p+1 \\ p+2 \\ \vdots \\ 2p \\ \vdots \\ hp+1 \\ hp+2 \\ \vdots \\ hp+p \end{pmatrix}$$

where $A(G)_k$ is the k the row of $A(G)$. For equation k above, $1 \le k \le p$, substitute equations $(p+k),(2p+k),\dots,(hp+k)$ back into equation (k), to get

$$A(G)_k (x_1,\dots,x_p)^T + \sum_{i=1}^{h} x_{ip+k} = \lambda x_k$$

$$\Rightarrow A(G)_k (x_1,\dots,x_p)^T + \frac{1}{\lambda}(x_k + x_k + \dots + x_k) = \lambda x_k$$

$$\Rightarrow A(G)_k (x_1,\dots,x_p)^T + \frac{h}{\lambda} x_k = \lambda x_k$$

$$\Rightarrow A(G)_k (x_1,\dots,x_p)^T = \left(\lambda - \frac{h}{\lambda}\right) x_k$$

Therefore $\left(\lambda - \frac{h}{\lambda}\right)$ is an eigenvalue of $A(G)$ corresponding to eigenvector $(x_1,\dots,x_p)$, and

$\left(\lambda - \frac{h}{\lambda}\right) = \alpha_j$, for some j, $1 \le j \le p$, where α_j is an eigenvalue of $A(G)$,

$$\Rightarrow \lambda^2 - \alpha_j \lambda - h = 0$$

$$\Rightarrow \{\lambda_{2j-1}, \lambda_{2j}\} = \frac{\alpha_j \pm \sqrt{\alpha_j^{\,2} + 4h}}{2}.$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $A(GSun(h,p))$ are $\lambda = 0$. □

Corollary 2.1

If $\alpha = +1$ is an eigenvalue of $A(G)$ for graph G, then

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ are two eigenvalue of $A(GSun(1, p))$, and these eigenvalues satisfy the golden ratio property.

If $\alpha = -1$ is an eigenvalue of $A(G)$ for graph G, then

$\lambda = \frac{-1 \pm \sqrt{5}}{2}$ are two eigenvalue of $A(GSun(1, p))$, and these eigenvalues satisfy the golden ratio property.

Proof

Now if $\alpha = 1$ is an eigenvalue of $A(G)$, then from Theorem 2.1,

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ are two eigenvalue of $A(GSun(1, p))$, and they satisfy the golden ratio property.

Now if $\alpha = -1$ is an eigenvalue of $A(G)$, then from Theorem 2.1,

$\lambda = \frac{-1 \pm \sqrt{5}}{2}$ are two eigenvalue of $A(GSun(1, p))$, and they satisfy the golden ratio property. □

Corollary 2.2

For the generalized sun graph $GSun(h, p)$,

$$\det[A(GSun(h,p))] = \det\begin{bmatrix} A(G_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix} = \begin{cases} 1 & \text{for } h = 1 \text{ and } p \text{ even} \\ -1 & \text{for } h = 1 \text{ and } p \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

Proof

$$A(GSun(h,p)) = \begin{bmatrix} A(G_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}, \text{ so}$$

$$\det[A(GSun(h,p))] = \det\begin{bmatrix} A(G_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

To determine the determinant of $A(GSun(h, p))$, expand along the $(p+1)th$ row of $A(GSun(h, p))$, with all entries equal to zero except for the $(p+1,1)th$ entry which is 1. Then $\det[A(CSun(h, p))] = (-1)^{p+2}\det[H_1]$ where $H_i = \det[A(GSun(h, p))]$ after deleting rows $p+j;\ 1 \le j \le i$ and deleting columns $k;\ 1 \le k \le i$.

Now expand along the $(p+1)th$ row of H_1, with all entries equal to zero except for the $(p+1,1)th$ entry which is 1. Then

$\det[A(GSun(h, p))] = (-1)^{p+2}(-1)^{p+2}\det[H_2] = (-1)^{(2(p+2))}\det[H_2]$.

Repeat for another $(p-2)$ iterations to get

$\det[A(GSun(h, p))] = (-1)^{(p(p+2))}\det[H_p] = (-1)^{(p^2+2p)}\det[H_p]$.

where $H_p = GSun(h, p)$ after deleting rows $p + j;\ 1 \le j \le p$ and deleting columns $k;\ 1 \le k \le p$, i.e.

$$H_p = \begin{bmatrix} I_{p,p} & I_{p,p} & \cdots & I_{p,p} \\ 0_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

Now if $h = 1$ then $H_p = I_{p,p}$ and $\det(H_p) = \det(I_{p,p}) = 1$.

Therefore, $\det[A(GSun(h, p))] = \begin{cases} 1 & for\ \ h = 1\ and\ \ p\ even \\ -1 & for\ \ h = 1\ and\ \ p\ odd \\ 0 & otherwise \end{cases}$ □

Theorem 2.2

The energy of the generalized sun graph $GSun(h, p)$ is

$E(GSun(h, p)) = \sum_{i=1}^{p} \sqrt{\alpha_j^2 + 4h}$ where α_j are the eigenvalues of $A(G)$.

Proof

$$E(GSun(h, p)) = \sum_{j=1}^{p(h+1)} |\lambda_j| \text{ where } \lambda_j \text{ are the eigenvalues of } A(GSun(h, p))$$

$$= \sum_{j=1}^{p} \left(\left| \frac{\alpha_j + \sqrt{\alpha_j^2 + 4h}}{2} \right| + \left| \frac{\alpha_j - \sqrt{\alpha_j^2 + 4h}}{2} \right| \right)$$

$$= \sum_{j=1}^{p} \left(\frac{\alpha_j + \sqrt{\alpha_j^2 + 4h}}{2} + \frac{-\alpha_j + \sqrt{\alpha_j^2 + 4h}}{2} \right)$$

$$= \sum_{j=1}^{p} \sqrt{\alpha_j^2 + 4h} \text{ where } \alpha_j \text{ are the eigenvalues of } A(G).$$ □

3. The Spectrum of the Complete Sun Graph

Let G be the complete sun graph $CompSun(h, p)$ which consists of the complete graph K_p, with h end vertices appended to each of the p vertices in K_p. Then $CompSun(h, p)$ has $n = (h+1)p$ vertices and $p\left(\frac{p-1}{2} + h\right)$ edges. Then the (nxn) adjacency matrix of $CompSun(h, p)$ is:

$$A(CompSun(h, p)) = \begin{bmatrix} A(K_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{n,n} & \cdots & 0_{n,n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{n,n} & \cdots & 0_{n,n} \end{bmatrix}.$$

Theorem 3.1

The eigenvalues of $CompSun(h, p)$ are

$\lambda = \dfrac{-1 \pm \sqrt{1 + 4h}}{2}$ with multiplicity $p - 1$,

$\lambda = \dfrac{(p-1) \pm \sqrt{(p-1)^2 + 4h}}{2}$ with multiplicity 1, and

$\lambda = 0$ with multiplicity $p(h-1)$.

Proof

The eigenvalues of $A(K_p)$ are

$\alpha = -1$ with multiplicity $p-1$, and

$\alpha = p-1$ with multiplicity 1.

See Jessop [4].

Therefore, from Theorem 2.1, the eigenvalues of $A(CompSun(h,p))$ are

$\lambda = \dfrac{-1 \pm \sqrt{1+4h}}{2}$ with multiplicity $p-1$

$\lambda = \dfrac{(p-1) \pm \sqrt{(p-1)^2 + 4h}}{2}$ with multiplicity 1.

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $A(CompSun(h,p))$ are 0. □

Corollary 3.1

There are $2(p-1)$ eigenvalues of $A(CompSun(1,p))$ which satisfy the golden ratio property.

Proof

As $\alpha = -1$ is an eigenvalue of $A(K_p)$ with multiplicity $p-1$, then by Corollary 2.1, $\lambda = \dfrac{-1 \pm \sqrt{5}}{2}$ are two eigenvalues of $A(CompSun(1,p))$, each with multiplicity $p-1$ which satisfy the golden ratio property. □

And from Corollary 2.2, we get

Corollary 3.2

For the complete sun graph $CompSun(h,p)$,

$$\det[A(CompSun(h,p))] = \det \begin{bmatrix} A(K_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix} = \begin{cases} 1 & for \ \ h=1 \ \ and \ \ n \ \ even \\ -1 & for \ \ h=1 \ \ and \ \ n \ \ odd \\ 0 & otherwise \end{cases}.$$

Theorem 3.2

The energy of the complete sun graph is

$$E(CompSun(h,p)) = (p-1)\sqrt{1+4h} + \sqrt{(p-1)^2 + 4h}.$$

Proof

From Theorem 2.2,

$$E(CompSun(h,p)) = \sum_{j=1}^{p} \sqrt{\alpha_j^{\,2} + 4h} \quad \text{where } \alpha_j \text{ are the eigenvalues of } A(K_p).$$

$$= (p-1)\sqrt{1+4h} + \sqrt{(p-1)^2 + 4h}.$$ □

4. The Spectrum of the Complete Split-Bipartite Sun Graph

Let G be the complete split-bipartite sun graph $BipSun(h,p)$ which consists of the complete split-bipartite graph $K_{\frac{p}{2},\frac{p}{2}}$, with h end vertices appended to each of the p vertices in $K_{\frac{p}{2},\frac{p}{2}}$. Then $BipSun(h,p)$ has $n = (h+1)p$ vertices and $\dfrac{p^2}{4} + ph$ edges. Then the (nxn) adjacency matrix of $BipSun(h,p)$ is:

$$A(BipSun(h,p)) = \begin{bmatrix} A\left(K_{\frac{p}{2},\frac{p}{2}}\right) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

Theorem 4.1

The eigenvalues of $A(BipSun(h,p))$ are

$$\lambda = \frac{p \pm \sqrt{p^2 + 16h}}{4} \text{ with multiplicity 1}$$

$$\lambda = \frac{-p \pm \sqrt{p^2 + 16h}}{4} \text{ with multiplicity 1}$$

$\lambda = \pm\sqrt{h}$ with multiplicity $p-2$, and

$\lambda = 0$ with multiplicity $p(h-1)$.

Proof

The eigenvalues of $A\left(K_{\frac{p}{2},\frac{p}{2}}\right)$ are

$\alpha = \pm\frac{p}{2}$ with multiplicity 1, and

$\alpha = 0$ with multiplicity $p-2$.

See Jessop [3].

Therefore, from Theorem 2.1, the eigenvalues of $A(BipSun(h,p))$ are

$$\lambda = \frac{\left(\frac{p}{2}\right) \pm \sqrt{\left(\frac{p}{2}\right)^2 + 4h}}{2} = \frac{p \pm \sqrt{p^2 + 16h}}{4} \text{ with multiplicity } 1$$

$$\lambda = \frac{\left(-\frac{p}{2}\right) \pm \sqrt{\left(-\frac{p}{2}\right)^2 + 4h}}{2} = \frac{-p \pm \sqrt{p^2 + 16h}}{4} \text{ with multiplicity } 1$$

$$\lambda = \frac{(0) \pm \sqrt{(0)^2 + 4h}}{2} = \pm\sqrt{h} \text{ with multiplicity } p-2.$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $A(BipSun(h,p))$ are 0. □

Corollary 4.1

Four eigenvalues of the adjacency matrix of the complete split-bipartite sun graph $BipSun(1,2)$ have the golden ratio property.

Proof

Setting $h = 1$ and $p = 2$ in Theorem 3.1, we have $\lambda = \frac{p \pm \sqrt{p^2 + 16h}}{4} = \frac{2 \pm \sqrt{2^2 + 16}}{4} = \frac{1 \pm \sqrt{5}}{2}$ with multiplicity 1 and $\lambda = \frac{-p \pm \sqrt{p^2 + 16h}}{4} = \frac{-1 \pm \sqrt{5}}{2}$ with multiplicity 1. These eigenvalues have the golden ratio property.

□

And from Corollary 2.2, we get

Corollary 4.2

For the complete split-bipartite sun graph $BipSun(h, p)$,

$$\det[A(BipSun(h, p))] = \begin{cases} 1 & for\ \ h = 1 \\ 0 & otherwise \end{cases}.$$

Theorem 4.2

The energy of the complete split-bipartite sun graph is

$$E(BipSun(h, p)) = \sqrt{p^2 + 16h} + 2(p-2)\sqrt{h}.$$

Proof

From Theorem 2.2,

$$E(BipSun(h, p)) = \sum_{j=1}^{p} \sqrt{\alpha_j^{\ 2} + 4h} \text{ where } \alpha_j \text{ are the eigenvalues of } A\left(K_{\frac{p}{2},\frac{p}{2}}\right).$$

$$= \sqrt{\left(\frac{p}{2}\right)^2 + 4h} + \sqrt{\left(-\frac{p}{2}\right)^2 + 4h} + (p-2)\sqrt{4h}.$$

$$= \sqrt{p^2 + 16h} + 2(p-2)\sqrt{h}.$$

□

5. The Spectrum of the Caterpillar Graph

Let G be the caterpillar graph $Caterpillar(h, p)$ (which is the same as the path sun graph), which consists of the path graph P_p, with h end vertices appended to each of the p vertices in P_p. Then $Caterpillar(h, p)$ has $n = (h+1)p$ vertices and $(p-1) + hp$ edges. Then the (nxn) adjacency matrix of $Caterpillar(h, p)$ is:

$$A(Caterpillar(h, p)) = \begin{bmatrix} A(P_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

Theorem 5.1

The eigenvalues of $Caterpillar(h, p)$ are

$$\{\lambda_{2k+1,2k+2}\} = \cos\left(\frac{\pi k}{p+1}\right) \pm \sqrt{\cos^2\left(\frac{\pi k}{p+1}\right) + h}, \text{ where } k = 1, \ldots, p, \text{ and}$$

$\lambda = 0$ with multiplicity $p(h-1)$.

Proof

The eigenvalues of $A(P_p)$ are

$$\alpha = 2\cos\left(\frac{\pi k}{p+1}\right),\ 1 \le k \le p.$$

See Jessop [3].

Therefore, from Theorem 2.1, the eigenvalues of $A(Caterpillar(h, p))$ are

$$\lambda = \frac{2\cos\left(\frac{\pi k}{p+1}\right) \pm \sqrt{\left(2\cos\left(\frac{\pi k}{p+1}\right)\right)^2 + 4h}}{2} = \cos\left(\frac{\pi k}{p+1}\right) \pm \sqrt{\cos^2\left(\frac{\pi k}{p+1}\right) + h},\ 1 \le k \le p.$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $A(Caterpillar(h, p))$ are 0. □

Corollary 5.1

Two of the eigenvalues of $A(Caterpillar(1,p))$ with $p = 3l - 1 \;\; l = 1,2,\ldots$, have the golden ratio property.

Proof

Let $p = 3l - 1 \;\; l = 1,2,\ldots,$. Then the eigenvalues of $A(P_{3l-1})$ are

$$\alpha = 2\cos\left(\frac{\pi k}{p+1}\right) = 2\cos\left(\frac{\pi k}{3l}\right) \;\; 1 \le k \le (3l-1), \text{ with multiplicity } 1.$$

Setting $k = l$, we get

$$\alpha = 2\cos\left(\frac{\pi l}{3l}\right) = 2\cos\left(\frac{\pi}{3}\right) = 1.$$

Then, by Corollary 2.1,

$\lambda = \dfrac{1 \pm \sqrt{5}}{2}$ are two eigenvalues of $A(Caterpillar(1,3l-1)) \;\; l = 1,2,\ldots$ which satisfy the golden ratio property.

□

And from Corollary 2.2, we get

Corollary 5.2

For the caterpillar graph $Caterpillar(h,p)$,

$$\det[A(Caterpillar(h,p))] = \begin{cases} 1 \;\; for \;\; h = 1 \;\; and \;\; p \;\; even \\ -1 \;\; for \;\; h = 1 \;\; and \;\; p \;\; odd \\ 0 \;\; otherwise \end{cases}.$$

Theorem 5.2

The energy of the caterpillar graph is

$$E(Caterpillar(h,p)) \qquad = 2\sum_{k=1}^{p}\sqrt{\cos^2\left(\frac{\pi k}{p+1}\right) + h}\,.$$

Proof

From Theorem 2.2,

$$E(Caterpillar(h,p)) \qquad = \sum_{j=1}^{p}\sqrt{\alpha_j^{\,2} + 4h} \;\; \text{where } \alpha_j \text{ are the eigenvalues of } A(P_p).$$

$$= \sum_{k=1}^{p}\sqrt{\left(2\cos\left(\frac{\pi k}{p+1}\right)\right)^2 + 4h}\,.$$

$$= 2\sum_{k=1}^{p}\sqrt{\cos^2\left(\frac{\pi k}{p+1}\right) + h}\,.$$

□

6. The Spectrum of the Cycle Sun Graph

Let G be the cycle sun graph $CycleSun(h,p)$ which consists of the cycle graph C_p, with h end vertices appended to each of the p vertices in C_p. Then $CycleSun(h,p)$ has $n = (h+1)p$ vertices and $(h+1)p$ edges. Then the (nxn) adjacency matrix of $CycleSun(h,p)$ is:

$$A(CycleSun(h,p)) = \begin{bmatrix} A(C_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

Theorem 6.1

The eigenvalues of $A(CycleSun(h,p))$ are

$$\lambda = \cos\left(\frac{2\pi k}{p}\right) \pm \sqrt{\cos^2\left(\frac{2\pi k}{p}\right) + h}$$, where $k = 0,1,\ldots, p-1$, and

$\lambda = 0$ with multiplicity $p(h-1)$.

Proof

The eigenvalues of $A(C_p)$ are

$$\alpha = 2\cos\left(\frac{2\pi k}{p}\right) \quad 0 \le k \le p-1.$$

See Jessop [3].

Therefore, from Theorem 2.1, the eigenvalues of $A(CycleSun(h,p))$ are

$$\lambda = \frac{2\cos\left(\frac{2\pi k}{p}\right) \pm \sqrt{\left(2\cos\left(\frac{2\pi k}{p}\right)\right)^2 + 4h}}{2} = \cos\left(\frac{2\pi k}{p}\right) \pm \sqrt{\cos^2\left(\frac{2\pi k}{p}\right) + h} \ ,0 \le k \le p-1.$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $A(CycleSun(h,p))$ are 0. □

Corollary 6.1

Two of the eigenvalues of $A(CycleSun(1,p))$ $p = 6l$, $l = 1,2,\ldots,$ satisfy the golden ratio property.

Proof

Let $p = 6l$ $l = 1,2,\ldots,$. Then the eigenvalues of $A(C_{6l})$ are

$$\alpha = 2\cos\left(\frac{2\pi k}{6l}\right) \quad 0 \le k \le 6l$$, with multiplicity 1.

Setting $k = l$, we get $\alpha = 2\cos\left(\frac{2\pi l}{6l}\right) = 1$.

Then, by Corollary 2.1,

$\lambda = \frac{1 \pm \sqrt{5}}{2}$ are two eigenvalues of $A(CycleSun(1,6l))$ $l = 1,2,\ldots$ which satisfy the golden ratio property.

□

And from Corollary 2.2, we get

Corollary 6.2

For the cycle sun graph $CycleSun(h,p)$,

$$\det[A(CycleSun(h,p))] = \begin{cases} 1 \ \ for \ \ h = 1 \ \ and \ \ p \ \ even \\ -1 \ \ for \ \ h = 1 \ \ and \ \ p \ \ odd \\ 0 \ \ otherwise \end{cases}.$$

Theorem 5.2

The energy of the cycle sun graph is

$$E(CycleSun(h,p)) = 2\sum_{k=0}^{p-1} \sqrt{\cos^2\left(\frac{2\pi k}{p}\right) + h}.$$

Proof

From Theorem 2.2,

$$E(CycleSun(h,p)) = \sum_{j=1}^{p} \sqrt{\alpha_j^{\,2} + 4h}$$ where α_j are the eigenvalues of $A(C_p)$.

$$= \sum_{k=0}^{p-1} \sqrt{\left(2\cos\left(\frac{2\pi k}{p}\right)\right)^2 + 4h} .$$

$$= 2\sum_{k=0}^{p-1} \sqrt{\cos^2\left(\frac{2\pi k}{p}\right) + h} . \qquad \square$$

7. The Spectrum of the Wheel Sun Graph

Let G be the wheel sun graph $WheelSun(h, p)$ which consists of the wheel graph W_p, with h end vertices appended to each of the p vertices in W_p. Then $WheelSun(h, p)$ has $n = (h+1)p$ vertices and $(h+2)(p-1)+h$ edges. Then the (nxn) adjacency matrix of $WheelSun(h, p)$ is:

$$A(WheelSun(h, p)) = \begin{bmatrix} A(W_p) & I_{p,p} & \cdots & I_{p,p} \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \\ \vdots & \vdots & \ddots & \vdots \\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p} \end{bmatrix}.$$

Theorem 7.1

The eigenvalues of $A(WheelSun(h, p))$ are

$\lambda = \cos\left(\frac{2\pi k}{p-1}\right) \pm \sqrt{\cos^2\left(\frac{2\pi k}{p-1}\right) + h}$ where $k = 1, \ldots, p-2$,

$\lambda = \frac{\left(1+\sqrt{p}\right) \pm \sqrt{\left(1+\sqrt{p}\right)^2 + 4h}}{2}$,

$\lambda = \frac{\left(1-\sqrt{p}\right) \pm \sqrt{\left(1-\sqrt{p}\right)^2 + 4h}}{2}$, and

$\lambda = 0$ with multiplicity $p(h-1)$. $\square$

Proof

The eigenvalues of $A(W_p)$ are

$\alpha = 2\cos\left(\frac{2\pi k}{p-1}\right)$ $1 \le k \le p-2$.

$\alpha = 1 \pm \sqrt{p}$, with multiplicity 1.

See Jessop [3].

Therefore, from Theorem 2.1, the eigenvalues of $A(WheelSun(h, p))$ are

$$\lambda = \frac{2\cos\left(\frac{2\pi k}{p-1}\right) \pm \sqrt{\left(2\cos\left(\frac{2\pi k}{p-1}\right)\right)^2 + 4h}}{2} = \cos\left(\frac{2\pi k}{p-1}\right) \pm \sqrt{\cos^2\left(\frac{2\pi k}{p-1}\right) + h} \ , 1 \le k \le p-2,$$

$\lambda = \frac{\left(1+\sqrt{p}\right) \pm \sqrt{\left(1+\sqrt{p}\right)^2 + 4h}}{2}$ with multiplicity 1, and

$\lambda = \frac{\left(1-\sqrt{p}\right) \pm \sqrt{\left(1-\sqrt{p}\right)^2 + 4h}}{2}$ with multiplicity 1.

The remaining $p(h+1)-2p=p(h-1)$ eigenvalues of $A(WheelSun(h,p))$ are 0. □

Corollary 7.1

Two of the eigenvalues of $A(WheelSun(1,p))$ $p=6l+1$, $l=1,2,...$ satisfy the golden ratio property.

Proof

Let $p=6l+1$. Then the eigenvalues of $A(W_{6l+1})$ are

$$\alpha=2\cos\left(\frac{2\pi k}{p-1}\right)=2\cos\left(\frac{2\pi k}{6l}\right)\quad 1\le k\le(3l-1).$$

Setting $k=l$, we get $\alpha=2\cos\left(\frac{2\pi l}{6l}\right)=2\cos\left(\frac{2\pi}{6}\right)=1$.

Then, by Corollary 2.1,

$\lambda=\frac{1\pm\sqrt{5}}{2}$ are two eigenvalues of $A(WheelSun(1,6l+1))$, $l=1,2,...$ which satisfy the golden ratio property.

□

And from Corollary 2.2, we get

Corollary 7.2

For the wheel sun graph $WheelSun(h,p)$,

$$\det[A(WheelSun(h,p))]=\begin{cases}1 & for\ \ h=1\ \ and\ \ p\ \ even\\ -1 & for\ \ h=1\ \ and\ \ p\ \ odd\\ 0 & otherwise\end{cases}.$$

Theorem 7.2

The energy of the wheel sun graph is

$$E(WheelSun(h,p))=2\sum_{k=1}^{p-2}\sqrt{\cos^2\left(\frac{2\pi k}{p-1}\right)+h}+\sqrt{\left(1+\sqrt{p}\right)^2+4h}+\sqrt{\left(1-\sqrt{p}\right)^2+4h}.$$

Proof

From Theorem 2.2,

$$E(WheelSun(h,p))=\sum_{i=1}^{p}\sqrt{\alpha_j^{\,2}+4h}\quad \text{where } \alpha_j \text{ are the eigenvalues of } A(W_p).$$

$$=\sum_{k=1}^{p-2}\sqrt{\left(2\cos\left(\frac{2\pi k}{p-1}\right)\right)^2+4h}+\sqrt{\left(1+\sqrt{p}\right)^2+4h}+\sqrt{\left(1-\sqrt{p}\right)^2+4h}.$$

$$=2\sum_{k=1}^{p-2}\sqrt{\cos^2\left(\frac{2\pi k}{p-1}\right)+h}+\sqrt{\left(1+\sqrt{p}\right)^2+4h}+\sqrt{\left(1-\sqrt{p}\right)^2+4h}.$$ □

8. The Spectrum of the Star Sun Graph

Let G be the star sun graph $StarSun(h,p)$ which consists of the star graph $S_{p-1,1}$, with h end vertices appended to each of the p vertices in $S_{p-1,1}$. Then $StarSun(h,p)$ has $n=(h+1)p$ vertices and $(h+1)(p-1)+h$ edges. Then the (nxn) adjacency matrix of $StarSun(h,p)$ is:

$$A(StarSun(h,p))=\begin{bmatrix}A(S_{(p-1,1)}) & I_{p,p} & \cdots & I_{p,p}\\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p}\\ \vdots & \vdots & \ddots & \vdots\\ I_{p,p} & 0_{p,p} & \cdots & 0_{p,p}\end{bmatrix}.$$

Theorem 8.1

The eigenvalues of $A(StarSun(h,p))$ are

$$\lambda = \frac{\sqrt{p-1} \pm \sqrt{p-1+4h}}{2} \text{ with multiplicity 1}$$

$$\lambda = \frac{-\sqrt{p-1} \pm \sqrt{p-1+4h}}{2} \text{ with multiplicity 1,}$$

$\lambda = \pm\sqrt{h}$ with multiplicity $(p-2)$, and

$\lambda = 0$ with multiplicity $p(h-1)$. □

Proof

The eigenvalues of $A(S_{p-1,1})$ are

$\alpha = \pm\sqrt{p-1}$, with multiplicity 1.

$\alpha = 0$, with multiplicity $p-2$.

See Jessop [3].

Therefore, from Theorem 2.1, the eigenvalues of $A(StarSun(h,p))$ are

$$\lambda = \frac{\sqrt{p-1} \pm \sqrt{\left(\sqrt{p-1}\right)^2 + 4h}}{2} = \frac{\sqrt{p-1} \pm \sqrt{p-1+4h}}{2}, \text{ with multiplicity } 1,$$

$$\lambda = \frac{-\sqrt{p-1} \pm \sqrt{\left(-\sqrt{p-1}\right)^2 + 4h}}{2} = \frac{-\sqrt{p-1} \pm \sqrt{p-1+4h}}{2}, \text{ with multiplicity } 1,$$

$$\lambda = \frac{0 \pm \sqrt{(0)^2 + 4h}}{2} = \pm\sqrt{h}, \text{ with multiplicity } p-2,$$

The remaining $p(h+1) - 2p = p(h-1)$ eigenvalues of $A(StarSun(h,p))$ are 0. □

Corollary 8.1

Four of the eigenvalues of $A(StarSun(1,2))$ satisfy the golden ratio property.

Proof

Setting $p = 2$ we get the eigenvalues of $A(S_{1,1})$ are

$\alpha = \pm\sqrt{2-1} = \pm 1$, with multiplicity 1.

Then, by Corollary 2.1,

$\lambda = \frac{\pm 1 \pm \sqrt{5}}{2}$ are four eigenvalues of $A(StarSun(1,p))$ which satisfy the golden ratio property. □

And from Corollary 2.2, we get

Corollary 8.1

For the star sun graph $StarSun(h,p)$

$$\det[A(StarSun(h,p))] = \begin{cases} 1 & \text{for } h = 1 \text{ and } p \text{ even} \\ -1 & \text{for } h = 1 \text{ and } p \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 8.2

The energy of the star sun graph is

$$E(StarSun(h,p)) \qquad = 2\sqrt{p-1+4h} + 2(p-2)\sqrt{h}.$$

Proof

From Theorem 2.2,

$$E(StarSun(h,p)) = \sum_{i=1}^{p}\sqrt{\alpha_j^{\,2}+4h} \text{ where } \alpha_j \text{ are the eigenvalues of } A(S_{p-1,1}).$$

$$= \sqrt{\left(+\sqrt{p-1}\right)^2+4h}+\sqrt{\left(-\sqrt{p-1}\right)^2+4h}+(p-2)\sqrt{(0)^2+4h}\,.$$

$$= \sqrt{p-1+4h}+\sqrt{p-1+4h}+(p-2)\sqrt{4h}$$

$$= 2\sqrt{p-1+4h}+2(p-2)\sqrt{h}\,.$$

9. Conclusion

In this paper, we determined the general solution for the spectra of graphs obtained by appending h end vertex to all vertices of a *base graph* and determined the energy of these graphs based on the eigenvalues of the base graph. We then determine the spectra and energy of such graphs where the base graph is the complete, complete split-bipartite, cycle, path, wheel and star graphs.

We also showed that the spectra of some of these graphs satisfy the golden ratio property for each of the classes of graphs above.

References

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