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On The Stability of Third Order Linear Autonomous Neutral Delay Differential Equations

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Abstract: A wide class of third order linear autonomous neutral delay differential equations with distributed type delays is considered. Anasymptotic result, a useful exponential estimate of the solutions, a stability criterion are established.

Keywords: Neutral delay differential equation; Characteristic equation; Asymptotic behavior; Solution; Stability.

1. Introduction

The theory of neutral delay differential equations is important both theoretical and practical interest. For the basic theory of neutral delay differential equations, the reader is referred to the books by Bellman and Cooke [1], Driver [2], El'sgol'ts and Norkin [3], Hale and Verduyn Lunel [4] Kolmanovski and Myshkis [5] and Lakshmikantham, *et al.* [6].

In this paper, we deal with the stability of the trivial solution for a third order linear autonomous neutral delay differential equation with constant delay. An asymptotic result for the solutions is obtained. Also, an estimate of the solutions is established. The sufficient conditions for the stability and the asymptotic stability of the trivial solution and some examples are given. Our results are derived by the use of real roots (with an appropriate property) of the corresponding (in a sense) characteristic equations.

The very interesting asymptotic and stability results were given by Philos and Purnaras [7-9]. The techniques applied in [10, 11] are originated in a combination of the methods used in [7-9].

Yeniçerioğlu [11] obtained some results on the qualitative behavior of the solutions of a second order linear autonomous delay differential equation with a single delay. The main idea in Yeniçerioğlu [11] is that of transforming the second order delay differential equation into a first order delay differential equation, by the use of a real root of the corresponding characteristic equation. The same idea will be used in this paper to obtain some general results.

Recently, Cahlon and Schmidt [12] have established the stability criteria for a third order delay differential equation. This equation is obtained the stability of third order delay differential equation using Pontryagin's theory for quasi-polynomials. However, we study the stability of the some problem using the method of characteristic roots.

Let us consider initial value problem for third orderneutral delay differential equation

$$\begin{aligned}\left[y''(t)+cy''(t-\tau)\right]' &= p_1y''(t)+p_2y''(t-\tau)+q_1y'(t)+q_2y'(t-\tau)\\ &\quad +v_1y(t)+v_2y(t-\tau)\ , \qquad t\geq 0, \end{aligned} \tag{1.1}$$

where $c, p_1, p_2, q_1, q_2, v_1, v_2$ are real numbers, τ is a positive real number. In a previous paper [13], we considered Eq. (1.1) with $= 0$, $q_2 = 0$ and $v_1 = 0$ which arose from a robotic model with damping and delay. There are no practical stability criteria of the zero solution of (1.1). For studies of stability of restricted special cases of (1.1) see [14, 15].

By a *solution* of the neutral delay differential equation (1.1), we mean a twice continuously differentiable real-valued function y defined on the interval $[-\tau,\infty)$, which is thrice continuously differentiable on $[0,\infty)$ and satisfies (1.1) for all $t\geq 0$.

Together with the neutral delay differential equation (1.1), it is customary to specify an *initial condition* of the form

$$y(t)=\phi(t) \quad \text{for} \quad -\tau\leq t\leq 0, \tag{1.2}$$

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where the initial function $\phi(t)$ is a given twice continuously differentiable real-valued function on the initial interval $[-\tau,0]$.

Equations (1.1) and (1.2) constitute an *initial value problem* (IVP, for short). It is known that, (see, for example, Driver [5]) for any given initial function ϕ, there exists a unique solution of the initial value problem (1.1) and (1.2) or, more briefly, the *solution* of the IVP (1.1) and (1.2).

Along with the neutral delay differential equation (1.1), we associate the equation

$$\lambda^3+\lambda^3 ce^{-\lambda\tau}=\lambda^2 p_1+\lambda q_1+v_1+e^{-\lambda\tau}\left(\lambda^2 p_2+\lambda q_2+v_2\right), \tag{1.3}$$

which will be called the *characteristic equation of* (1.1). Eq. (1.3) is obtained from (1.1) by looking for solutions of the form $y(t)=e^{\lambda t}$ for $t\geq-\tau$.

For a given real root λ_0 of the characteristic equation (1.3), we consider the (second order) neutral delay differential equation

$$\begin{aligned} z''(t)+ce^{-\lambda_0\tau}z''(t-\tau)&=(p_1-3\lambda_0)z'(t)+e^{-\lambda_0\tau}(p_2-3c\lambda_0)z'(t-\tau)+(q_1+2p_1\lambda_0-3\lambda_0^2)z(t)\\ &+e^{-\lambda_0\tau}(q_2+2p_2\lambda_0-3c\lambda_0^2)z(t-\tau)-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_{t-\tau}^{t}z(s)ds\,. \end{aligned} \tag{1.4}$$

A *solution* of the neutral delay differential equation (1.4) is a continuous real-valued function z defined on the interval $[-\tau,\infty)$, which is twice continuously differentiable on $[0,\infty)$ and satisfies (1.4) for all $t\geq 0$.

With the neutral delay differential equation (1.4), we associate the equation

$$\begin{aligned} \delta^2+c\delta^2 e^{-(\lambda_0+\delta)\tau}&=(p_1-3\lambda_0)\delta+(p_2-3c\lambda_0)\delta e^{-(\lambda_0+\delta)\tau}+q_1+2p_1\lambda_0-3\lambda_0^2\\ &+(q_2+2p_2\lambda_0-3c\lambda_0^2)e^{-(\lambda_0+\delta)\tau}-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\delta^{-1}(1-e^{-\delta\tau}), \end{aligned} \tag{1.5}$$

which is said to be the *characteristic equation of* (1.4). This equation is obtained from (1.4) by seeking solutions of the form $z(t)=e^{\delta t}$ for $t\geq-\tau$.

For our convenience, we introduce some notations. For a given real root λ_0 of the characteristic equation (1.3), we set

$$\beta_{\lambda_0}=e^{-\lambda_0\tau}\left((p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\tau-q_2-2p_2\lambda_0+3c\lambda_0^2\right)-q_1-2p_1\lambda_0+3\lambda_0^2 \tag{1.6}$$

and, also, we define

$$\begin{aligned} L(\lambda_0;\phi)&=\phi''(0)+c\phi''(-\tau)+(\lambda_0-p_1)\phi'(0)+(\lambda_0 c-p_2)\phi'(-\tau)+(\lambda_0^2-p_1\lambda_0-q_1)\phi(0)\\ &+(c\lambda_0^2-p_2\lambda_0-q_2)\phi(-\tau)+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_{-\tau}^{0}e^{-\lambda_0 s}\phi(s)ds\,; \end{aligned} \tag{1.7}$$

in addition, provided that $\beta_{\lambda_0}\neq 0$, we define

$$\Phi_1(\lambda_0;\phi)(t)=\phi(t)e^{-\lambda_0 t}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} \qquad \text{for} \quad -\tau\leq t\leq 0. \tag{1.8}$$

We will now give a proposition, which plays a crucial role in obtaining our main results.

Proposition 1.1. *Let* λ_0 *be real root of the characteristic equation* (1.3), *and let* β_{λ_0} *and* $L(\lambda_0;\phi)$ *be defined by* (1.6) *and* (1.7), *respectively. Suppose that* $\beta_{\lambda_0}\neq 0$, *and define* $\Phi_1(\lambda_0;\phi)$ *by* (1.8).

Then a continuous real-valued function y *defined on the interval* $[-\tau,\infty)$ *is the solution of the IVP* (1.1) *and* (1.2) *if and only if the function* z *defined by*

$$z(t)=y(t)e^{-\lambda_0 t}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} \qquad \textit{for} \quad t\geq-\tau \tag{1.9}$$

is the solution of the neutral delay differential equation (1.4) *which satisfies the initial condition*

$$z(t)=\Phi_1(\lambda_0;\phi)(t) \qquad \textit{for} \quad -\tau\leq t\leq 0. \tag{1.10}$$

Proof. Let y be the solution of the IVP (1.1) and (1.2).Define

$$x(t)=e^{-\lambda_0 t}y(t) \qquad \text{for} \quad t\in[-\tau,\infty), \tag{1.11}$$

where λ_0 is a real root of the characteristic equation (1.3). Then, for every $t\geq 0$,we have

$$\Big[x''(t)+ce^{-\lambda_0\tau}x''(t-\tau)+(3\lambda_0-p_1)x'(t)+e^{-\lambda_0\tau}(3c\lambda_0-p_2)x'(t-\tau)$$

$$\begin{aligned}&+(3\lambda_0^2-2p_1\lambda_0-q_1)x(t)+e^{-\lambda_0\tau}(3c\lambda_0^2-2p_2\lambda_0-q_2)x(t-\tau)\big]'\\&=(p_1\lambda_0^2+q_1\lambda_0+v_1-\lambda_0^3)x(t)+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)x(t-\tau).\end{aligned}\qquad(1.12)$$

Moreover, the initial condition (1.2) can be equivalently written

$$x(t)=e^{-\lambda_0 t}\phi(t)\quad\text{for}\quad t\in[-\tau,0].\qquad(1.13)$$

Furthermore, by using (1.3) and taking into account (1.13), we can verify that (1.12) is equivalent to

$$\begin{aligned}&x''(t)+ce^{-\lambda_0\tau}x''(t-\tau)+(3\lambda_0-p_1)x'(t)+e^{-\lambda_0\tau}(3c\lambda_0-p_2)x'(t-\tau)+(3\lambda_0^2-2p_1\lambda_0-q_1)x(t)\\&\quad+e^{-\lambda_0\tau}(3c\lambda_0^2-2p_2\lambda_0-q_2)x(t-\tau)=(p_1\lambda_0^2+q_1\lambda_0+v_1-\lambda_0^3)\int_0^t x(s)ds\\&+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^t x(s-\tau)ds+x''(0)+ce^{-\lambda_0\tau}x''(-\tau)+(3\lambda_0-p_1)x'(0)\\&+e^{-\lambda_0\tau}(3c\lambda_0-p_2)x'(-\tau)+(3\lambda_0^2-2p_1\lambda_0-q_1)x(0)+e^{-\lambda_0\tau}(3c\lambda_0^2-2p_2\lambda_0-q_2)x(-\tau),\end{aligned}$$

$$\begin{aligned}&x''(t)+ce^{-\lambda_0\tau}x''(t-\tau)=(p_1-3\lambda_0)x'(t)+e^{-\lambda_0\tau}(p_2-3c\lambda_0)x'(t-\tau)+(q_1+2p_1\lambda_0-3\lambda_0^2)x(t)\\&\quad+e^{-\lambda_0\tau}(q_2+2p_2\lambda_0-3c\lambda_0^2)x(t-\tau)+(p_1\lambda_0^2+q_1\lambda_0+v_1-\lambda_0^3)\int_0^t x(s)ds\\&\quad+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_{-\tau}^{t-\tau} x(s)ds+\phi''(0)-2\lambda_0\phi'(0)+\lambda_0^2\phi(0)\\&\quad+c(\phi''(-\tau)-2\lambda_0\phi'(-\tau)+\lambda_0^2\phi(-\tau))+(3\lambda_0-p_1)(\phi'(0)-\lambda_0\phi(0))\\&\quad+(3c\lambda_0-p_2)(\phi'(-\tau)-\lambda_0\phi(-\tau))+(3\lambda_0^2-2p_1\lambda_0-q_1)\phi(0)\\&\quad+(3c\lambda_0^2-2p_2\lambda_0-q_2)\phi(-\tau),\end{aligned}$$

$$\begin{aligned}&x''(t)+ce^{-\lambda_0\tau}x''(t-\tau)=(p_1-3\lambda_0)x'(t)+e^{-\lambda_0\tau}(p_2-3c\lambda_0)x'(t-\tau)+(q_1+2p_1\lambda_0-3\lambda_0^2)x(t)\\&\quad+e^{-\lambda_0\tau}(q_2+2p_2\lambda_0-3c\lambda_0^2)x(t-\tau)+(p_1\lambda_0^2+q_1\lambda_0+v_1-\lambda_0^3)\int_0^t x(s)ds\\&\quad+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{t-\tau} x(s)ds+L(\lambda_0;\phi),\end{aligned}$$

$$\begin{aligned}&x''(t)+ce^{-\lambda_0\tau}x''(t-\tau)=(p_1-3\lambda_0)x'(t)+e^{-\lambda_0\tau}(p_2-3c\lambda_0)x'(t-\tau)+(q_1+2p_1\lambda_0-3\lambda_0^2)x(t)\\&\quad+e^{-\lambda_0\tau}(q_2+2p_2\lambda_0-3c\lambda_0^2)x(t-\tau)-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^t x(s)ds\\&\quad+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{t-\tau} x(s)ds+L(\lambda_0;\phi),\end{aligned}$$

$$\begin{aligned}&x''(t)+ce^{-\lambda_0\tau}x''(t-\tau)=(p_1-3\lambda_0)x'(t)+e^{-\lambda_0\tau}(p_2-3c\lambda_0)x'(t-\tau)+(q_1+2p_1\lambda_0-3\lambda_0^2)x(t)\\&+e^{-\lambda_0\tau}(q_2+2p_2\lambda_0-3c\lambda_0^2)x(t-\tau)-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_{t-\tau}^t x(s)ds+L(\lambda_0;\phi),\end{aligned}\qquad(1.14)$$

where $L(\lambda_0;\phi)$ was given in (1.7).

Now, we take into account the assumption $\beta_{\lambda_0}\neq 0$ and we define

$$z(t)=x(t)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\quad\text{for}\quad t\geq -r.\qquad(1.15)$$

Then, because of definition of β_{λ_0} by (1.6), it is a matter of elementary calculations to show that x satisfies (1.14) for $t\geq 0$ if and only if z satisfies (1.4) for $t\geq 0$, i.e., if and only if z is a solution of the neutral delay differential equation (1.4). Moreover, we see that the initial condition (1.13) is equivalently written as follows

$$z(t)=e^{-\lambda_0 t}\phi(t)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} \quad \text{for} \quad -\tau\le t\le 0. \tag{1.16}$$

We have thus proved that y is the solution of the IVP (1.1) and (1.2) if and only if z is the solution of the neutral delay differential equation (1.4) which satisfies the initial condition (1.16). By (1.11), we see that (1.15)coincides with (1.9). Also, by taking into account the definition of $\Phi_1(\lambda_0;\phi)$ by (1.8), we observe that (1.16) coincides with the initial condition (1.10). The proof of the proposition1.1 is completed.

For a given real root λ_0 and δ_0 of the characteristic equation (1.3) and (1.5), respectively, we consider the (first order) neutral delay differential equation

$$\begin{aligned} w'(t)+ce^{-(\lambda_0+\delta_0)\tau}w'(t-\tau)&=(p_1-3\lambda_0-2\delta_0)w(t)+e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)w(t-\tau)\\ &+e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2-(p_2-3c\lambda_0)\delta_0-(q_2+2p_2\lambda_0-3c\lambda_0^2)\right)\int_{t-\tau}^{t}w(s)ds\\ &+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau}e^{-\delta_0 s}\left\{\int_{t-s}^{t}w(u)du\right\}ds. \end{aligned} \tag{1.17}$$

By a *solution* of the neutral delay (1.17), we mean a continuous real-valued function w defined on the interval $[-\tau,\infty)$, which is continuously differentiable on $[0,\infty)$ and satisfies (1.17) for all $t\ge 0$.

The *characteristic equation* of the neutral delay differential equation (1.17) is

$$\begin{aligned} \gamma\left(1+ce^{-(\lambda_0+\delta_0+\gamma)\tau}\right)&=p_1-3\lambda_0-2\delta_0+\left(p_2-3c\lambda_0-2c\delta_0\right)e^{-(\lambda_0+\delta_0+\gamma)\tau}\\ &+e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right)\gamma^{-1}\left(1-e^{-\gamma\tau}\right)\\ &+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\,\gamma^{-1}\left\{\delta_0^{-1}\left(1-e^{-\delta_0\tau}\right)-(\delta_0+\gamma)^{-1}\left(1-e^{-(\delta_0+\gamma)\tau}\right)\right\}. \end{aligned} \tag{1.18}$$

The last equation is obtained from (1.17) by seeking solutions of the form $w(t)=e^{\gamma t}$ for $t\ge-\tau$.

For our convenience, we introduce some notations. For a given real root λ_0 of the characteristic equation (1.3) and a given real root δ_0 of the characteristic equation (1.5), we set

$$\begin{aligned} \eta_{\lambda_0,\delta_0}&=e^{-(\lambda_0+\delta_0)\tau}\left\{\left(p_2\delta_0-3c\lambda_0\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2\right)\tau-p_2+3c\lambda_0+2c\delta_0\right\}\\ &-p_1+3\lambda_0+2\delta_0-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau}s\,e^{-\delta_0 s}ds \end{aligned} \tag{1.19}$$

and let $\Phi_1(\lambda_0;\phi)$ be defined by (1.8). Also, we define

$$\begin{aligned} R(\lambda_0,\delta_0;\phi)&=\left(\Phi_1(\lambda_0;\phi)\right)'(0)-\delta_0\Phi_1(\lambda_0;\phi)(0)\\ &+ce^{-\lambda_0\tau}\left(\left(\Phi_1(\lambda_0;\phi)\right)'(-\tau)-\delta_0\Phi_1(\lambda_0;\phi)(-\tau)\right)-(p_1-3\lambda_0-2\delta_0)\Phi_1(\lambda_0;\phi)(0)\\ &-e^{-\lambda_0\tau}(p_2-3c\lambda_0-2c\delta_0)\Phi_1(\lambda_0;\phi)(-\tau)\\ &+e^{-(\lambda_0+\delta_0)\tau}\left((p_2-3c\lambda_0)\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2\right)\int_{-\tau}^{0}e^{-\delta_0 s}\Phi_1(\lambda_0;\phi)(s)ds\\ &-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau}e^{-\delta_0 s}\left\{\int_{-s}^{0}e^{-\delta_0 u}\Phi_1(\lambda_0;\phi)(u)du\right\}ds, \end{aligned} \tag{1.20}$$

where $\left(\Phi_1(\lambda_0;\phi)\right)'$ is derivative of $\Phi_1(\lambda_0;\phi)$; in addition, provided that $\eta_{\lambda_0,\delta_0}\ne 0$, we define

$$\Phi_2(\lambda_0,\delta_0;\phi)(t)=e^{-\delta_0 t}\Phi_1(\lambda_0;\phi)(t)-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}} \quad \text{for} \quad t\in[-\tau,0]. \tag{1.21}$$

We will now give a proposition, which plays a crucial role in obtaining our main results.

Proposition 1.2. *Let* δ_0 *be real root of the characteristic equation* (1.5), *and let* $\eta_{\lambda_0,\delta_0}$ *and* $R(\lambda_0,\delta_0;\phi)$ *be defined by* (1.19) *and* (1.20), *respectively. Suppose that* $\eta_{\lambda_0,\delta_0}\ne 0$ *and define* $\Phi_2(\lambda_0,\delta_0;\phi)$ by (1.21).

Then a continuous real-valued function z *defined on the interval* $[-\tau,\infty)$ *is the solution of the neutral delay differential equation* (1.4) *which satisfies the initial condition* (1.10) *if and only if the function* w *defined by*

$$w(t)=e^{-\delta_0 t}z(t)-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}} \qquad \textit{for } t\geq -\tau \tag{1.22}$$

is the solution of the neutral delay differential equation (1.17) *which satisfies the initial condition*

$$w(t)=\Phi_2(\lambda_0,\delta_0;\phi)(t) \qquad \textit{for} \quad -\tau\leq t\leq 0. \tag{1.23}$$

Proof. Let now z be the solution of (1.4) and (1.10) and δ_0 be a real root of the characteristic equation (1.5). Define

$$v(t)=e^{-\delta_0 t}z(t) \qquad \text{for all } t\in[-\tau,\infty), \tag{1.24}$$

where δ_0 is a real root of the characteristic equation (1.5). Then, for every $t\geq 0$,we have

$$\begin{aligned}
v''(t)+ce^{-(\lambda_0+\delta_0)\tau}v''(t-\tau)&=(p_1-3\lambda_0-2\delta_0)v'(t)+e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)v'(t-\tau)\\
&+(\delta_0 p_1-3\lambda_0\delta_0+q_1+2p_1\lambda_0-3\lambda_0^2-\delta_0^2)v(t)\\
&+e^{-(\lambda_0+\delta_0)\tau}(\delta_0 p_2-3c\lambda_0\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2)v(t-\tau)\\
&-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^\tau e^{-\delta_0 s}v(t-s)ds.
\end{aligned} \tag{1.25}$$

Moreover, the initial condition (1.16) can be equivalently written

$$v(t)=e^{-\delta_0 t}\Phi_1(\lambda_0;\phi)(t) \qquad \text{for} \quad t\in[-\tau,0], \tag{1.26}$$

where Φ_1 was given in (1.8). Furthermore, by using (1.5) and taking into account (1.26), we can verify that (1.25) is equivalent to

$$\begin{aligned}
v'(t)+ce^{-(\lambda_0+\delta_0)\tau}v'(t-\tau)&=v'(0)+ce^{-(\lambda_0+\delta_0)\tau}v'(-\tau)+(p_1-3\lambda_0-2\delta_0)v(t)\\
&-(p_1-3\lambda_0-2\delta_0)v(0)+e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)v(t-\tau)\\
-e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)v(-\tau)&+(\delta_0 p_1-3\lambda_0\delta_0+q_1+2p_1\lambda_0-3\lambda_0^2-\delta_0^2)\int_0^t v(s)ds\\
&+e^{-(\lambda_0+\delta_0)\tau}(\delta_0 p_2-3c\lambda_0\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2)\int_0^t v(s-\tau)ds\\
&-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^\tau e^{-\delta_0 s}\left\{\int_0^t v(u-s)du\right\}ds,
\end{aligned}$$

$$\begin{aligned}
&v'(t)+ce^{-(\lambda_0+\delta_0)\tau}v'(t-\tau)=\left(\Phi_1(\lambda_0;\phi)\right)'(0)-\delta_0\Phi_1(\lambda_0;\phi)(0)\\
&+ce^{-\lambda_0\tau}\left(\left(\Phi_1(\lambda_0;\phi)\right)'(-\tau)-\delta_0\Phi_1(\lambda_0;\phi)(-\tau)\right)+(p_1-3\lambda_0-2\delta_0)v(t)\\
&\qquad-(p_1-3\lambda_0-2\delta_0)\Phi_1(\lambda_0;\phi)(0)+e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)v(t-\tau)\\
&\qquad-e^{-\lambda_0\tau}(p_2-3c\lambda_0-2c\delta_0)\Phi_1(\lambda_0;\phi)(-\tau)\\
&\qquad+(\delta_0 p_1-3\lambda_0\delta_0+q_1+2p_1\lambda_0-3\lambda_0^2-\delta_0^2)\int_0^t v(s)ds\\
&\qquad+e^{-(\lambda_0+\delta_0)\tau}(\delta_0 p_2-3c\lambda_0\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2)\left\{\int_{-\tau}^0 v(s)ds+\int_0^{t-\tau}v(s)ds\right\}\\
&\qquad-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^\tau e^{-\delta_0 s}\left\{\int_{-s}^0 v(u)du+\int_0^{t-s}v(u)du\right\}ds,
\end{aligned}$$

$$\begin{aligned}
&v'(t)+ce^{-(\lambda_0+\delta_0)\tau}v'(t-\tau)=(p_1-3\lambda_0-2\delta_0)v(t)+e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)v(t-\tau)\\
&\qquad+\left\{e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2-(p_2-3c\lambda_0)\delta_0-(q_2+2p_2\lambda_0-3c\lambda_0^2)\right)\right.
\end{aligned}$$

$$+\delta_0^{-1}\left(1-e^{-\delta_0\tau}\right)\left(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right)e^{-\lambda_0\tau}\ \Big\}\int_0^t v(s)ds$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left(\delta_0(p_2-3c\lambda_0)+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2\right)\int_0^{t-\tau} v(s)ds$$

$$-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{t-s} v(u)du\right\}ds+R(\lambda_0,\delta_0;\phi)\,,$$

$$v'(t)+ce^{-(\lambda_0+\delta_0)\tau}v'(t-\tau)=(p_1-3\lambda_0-2\delta_0)v(t)+e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)v(t-\tau)$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2-(p_2-3c\lambda_0)\delta_0-(q_2+2p_2\lambda_0-3c\lambda_0^2)\right)\int_{t-\tau}^{t} v(s)ds$$

$$+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau} e^{-\delta_0 s}ds\int_0^{t} v(s)ds$$

$$-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{t-s} v(u)du\right\}ds+R(\lambda_0,\delta_0;\phi)\,,$$

$$v'(t)+ce^{-(\lambda_0+\delta_0)\tau}v'(t-\tau)=(p_1-3\lambda_0-2\delta_0)v(t)+e^{-(\lambda_0+\delta_0)\tau}(p_2-3c\lambda_0-2c\delta_0)v(t-\tau)$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2-(p_2-3c\lambda_0)\delta_0-(q_2+2p_2\lambda_0-3c\lambda_0^2)\right)\int_{t-\tau}^{t} v(s)ds$$

$$+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau} e^{-\delta_0 s}\left\{\int_{t-s}^{t} v(u)du\right\}ds+R(\lambda_0,\delta_0;\phi)\,, \tag{1.27}$$

where $R(\lambda_0,\delta_0;\phi)$ was given in (1.20).

Next, we take into account the assumption $\eta_{\lambda_0,\delta_0}\neq 0$ and we define

$$w(t)=v(t)-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}} \qquad \text{for } t\geq -\tau\,. \tag{1.28}$$

Then, because of definition of $\eta_{\lambda_0,\delta_0}$ by (1.19), it is a matter of elementary calculations to show that v satisfies (1.27) for $t\geq 0$ if and only if w satisfies (1.17) for $t\geq 0$, i.e., if and only if w is a solution of the neutral delay differential equation (1.17). Moreover, we see that the initial condition (1.26) is equivalently written as follows

$$w(t)=e^{-\delta_0 t}\Phi_1(\lambda_0;\phi)(t)-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}} \qquad \text{for} \qquad -\tau\leq t\leq 0\,. \tag{1.29}$$

We have thus proved that z is the solution of (1.4) and (1.10) if and only if w is the solution of the neutral delay differential equation (1.17) which satisfies the initial condition (1.29). By (1.24), we see that (1.28) coincides with (1.22). Also, by taking into account the definition of $\Phi_2(\lambda_0,\delta_0;\phi)$ by (1.21), we observe that (1.29) coincides with the initial condition (1.23). The proof of the proposition 1.2 is completed.

Let $C([-\tau,0],IR)$ be the Banach space of all continuous real-valued functions on the interval $[-\tau,0]$, endowed with the usual sup-norm

$$\|\Psi\|=\max_{-\tau\leq t\leq 0}|\Psi(t)| \qquad \text{for} \quad \Psi\in C([-\tau,0],IR).$$

Moreover, let $C^2([-\tau,0],IR)$ be the set of all twice continuously differentiable real-valued functions on the interval $[-\tau,0]$. This set is a Banach space with the norm

$$\|\Psi\|_{C^2}=\max\left\{\|\Psi\|\ ,\ \|\Psi'\|\ ,\ \|\Psi''\|\right\} \qquad \text{for} \quad \Psi\in C^2([-\tau,0],IR).$$

As it concerns the IVP (1.1) and (1.2) studied in this paper, the initial function ϕ belongs to $C^2([-\tau,0],IR)$. So, the notation $\|\phi\|_{C^2}$ used in Section 3 is defined by

$$\|\phi\|_{C^2}=\max\left\{\|\phi\|\ ,\ \|\phi'\|\ ,\ \|\phi''\|\right\}\equiv\max\left\{\max_{-\tau\leq t\leq 0}|\phi(t)|\ ,\ \max_{-\tau\leq t\leq 0}|\phi'(t)|\ ,\ \max_{-\tau\leq t\leq 0}|\phi''(t)|\ \right\}.$$

Before closing this section, we will give three well-known definitions. The trivial solution of theneutral delay differential equation (1.1) is said to be "stable" (at 0) if for every $\varepsilon > 0$, there exists a $\ell = \ell(\varepsilon) > 0$ such that, for any $\phi \in C^2([-\tau,0], IR)$ with $\|\phi\|_{C^2} < \ell$, the solution y of the IVP (1.1) and (1.2) satisfies

$$|y(t)| < \varepsilon \quad \text{for all } t \in [-\tau,\infty).$$

Otherwise, the trivial solution of (1.1) is said to be "unstable" (at 0). Moreover, the trivial solution of (1.1) is called "asymptotically stable" (at 0) if it is stable in the above sense and, in addition, there exists a $\ell_0 > 0$ such that, for any $\phi \in C^2([-\tau,0], IR)$ with $\|\phi\|_{C^2} < \ell_0$, the solution y of the IVP (1.1)-(1.2) satisfies

$$\lim_{t\to\infty} |y(t)| = 0; \quad \text{i.e.,} \quad \lim_{t\to\infty} y(t) = 0.$$

2. An Asymptotic Results

Our purpose in this section is to establish the following theorem.

Theorem 2.1. *Let* λ_0 *be real root of the characteristic equation* (1.3), *and let* β_{λ_0} *and* $L(\lambda_0;\phi)$ *be defined by* (1.6) and (1.7), *respectively. Furthermore, let* δ_0 *be real root of the characteristic equation* (1.5), *and let* $\eta_{\lambda_0,\delta_0}$, $R(\lambda_0,\delta_0;\phi)$ *and* $\Phi_2(\lambda_0,\delta_0;\phi)$ *be defined by* (1.19), (1.20) *and* (1.21), *respectively. Moreover, let* γ_0 *be a real root of the characteristic equation* (1.18). *Suppose that* $\beta_{\lambda_0} \neq 0$ *and* $\eta_{\lambda_0,\delta_0} \neq 0$. *(Note that, because of* $\beta_{\lambda_0} \neq 0$, *we always have* $\delta_0 \neq 0$ *and* $\gamma_0 \neq -\delta_0$. *Furthermore, because of* $\eta_{\lambda_0,\delta_0} \neq 0$, *we always have* $\gamma_0 \neq 0$. *) Set*

$$\begin{aligned}\xi_{\lambda_0,\delta_0,\gamma_0} &= e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left[(p_2 - 3c\lambda_0 - 2c\delta_0 - c\gamma_0)\tau + c\right] \\ &+ e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right)\int_0^\tau e^{-\gamma_0 s} s\,ds \\ &+ e^{-\lambda_0\tau}\left(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3\right)\int_0^\tau e^{-\delta_0 s}\left\{\int_0^s e^{-\gamma_0 u} u\,du\right\}ds \qquad (2.1)\end{aligned}$$

and, also, define

$$\begin{aligned}K(\lambda_0,\delta_0,\gamma_0;\phi) &= \Phi_2(\lambda_0,\delta_0;\phi)(0) + ce^{-(\lambda_0+\delta_0)\tau}\Phi_2(\lambda_0,\delta_0;\phi)(-\tau) \\ &+ e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left(p_2 - 3c\lambda_0 - 2c\delta_0 - c\gamma_0\right)\int_{-\tau}^0 e^{-\gamma_0 s}\Phi_2(\lambda_0,\delta_0;\phi)(s)ds \\ &+ e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right)\int_0^\tau e^{-\gamma_0 s}\left\{\int_{-s}^0 e^{-\gamma_0 u}\Phi_2(\lambda_0,\delta_0;\phi)(u)du\right\}ds \\ &+ e^{-\lambda_0\tau}\left(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3\right)\int_0^\tau e^{-\delta_0 s}\left\{\int_0^s e^{-\gamma_0 u}\left\{\int_{-u}^0 e^{-\gamma_0\omega}\Phi_2(\lambda_0,\delta_0;\phi)(\omega)d\omega\right\}du\right\}ds. \qquad (2.2)\end{aligned}$$

Assume that

$$\begin{aligned}\mu_{\lambda_0,\delta_0,\gamma_0} &= e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left[\left|p_2 - 3c\lambda_0 - 2c\delta_0 - c\gamma_0\right|\tau + |c|\right] \\ &+ e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right|\int_0^\tau e^{-\gamma_0 s} s\,ds \\ &+ e^{-\lambda_0\tau}\left|p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3\right|\int_0^\tau e^{-\delta_0 s}\left\{\int_0^s e^{-\gamma_0 u} u\,du\right\}ds < 1. \qquad (2.3)\end{aligned}$$

(This assumption guarantees that $1 + \xi_{\lambda_0,\delta_0,\gamma_0} > 0$. *) Then the solution* y *of the IVP* (1.1) *and* (1.2) *satisfies*

$$\lim_{t\to\infty}\left\{e^{-\gamma_0 t}\left[e^{-(\lambda_0+\delta_0)t}y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t} - \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right]\right\} = \frac{K(\lambda_0,\delta_0,\gamma_0;\phi)}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}. \qquad (2.4)$$

Before we prove the above theorem, we will present some observations, which are concerned with a real root λ_0 of the characteristic equation (1.3), a real root δ_0 of the characteristic equation (1.5) and a real root γ_0 of the characteristic equation (1.18).

Let $F(\delta)$ denote the characteristic function of (1.5), i.e.,

$$\begin{aligned}F(\delta) = {} & \delta^2 + c\delta^2 e^{-(\lambda_0+\delta)\tau} - (p_1 - 3\lambda_0)\delta - (p_2 - 3c\lambda_0)\,\delta\, e^{-(\lambda_0+\delta)\tau} - q_1 - 2p_1\lambda_0 + 3\lambda_0^2 \\ & - (q_2 + 2p_2\lambda_0 - 3c\lambda_0^2)e^{-(\lambda_0+\delta)\tau} + e^{-\lambda_0\tau}(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\,\delta^{-1}(1 - e^{-\delta\tau})\,.\end{aligned}$$

Since $\delta = 0$ is a removable singularity of $F(\delta)$, we can regard $F(\delta)$ as a entire function with

$$F(0) = e^{-\lambda_0\tau}\left((p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\tau - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right) - q_1 - 2p_1\lambda_0 + 3\lambda_0^2 \equiv \beta_{\lambda_0}\,.$$

Hence, if we assume that $\beta_{\lambda_0} \neq 0$, then we always have $\delta_0 \neq 0$. Let $G(\gamma)$ denote the characteristic function of (1.18), i.e.,

$$\begin{aligned}G(\gamma) = {} & -\gamma\left(1 + ce^{-(\lambda_0+\delta_0+\gamma)\tau}\right) + p_1 - 3\lambda_0 - 2\delta_0 + \left(p_2 - 3c\lambda_0 - 2c\delta_0\right)e^{-(\lambda_0+\delta_0+\gamma)\tau} \\ & + e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right)\gamma^{-1}\left(1 - e^{-\gamma\tau}\right) \\ & + e^{-\lambda_0\tau}(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\,\gamma^{-1}\left\{\delta_0^{-1}\left(1 - e^{-\delta_0\tau}\right) - (\delta_0 + \gamma)^{-1}\left(1 - e^{-(\delta_0+\gamma)\tau}\right)\right\}.\end{aligned}$$

Since $\gamma = 0$ is a removable singularity of $G(\gamma)$, we can regard $G(\gamma)$ as a entire function with

$$\begin{aligned}G(0) = {} & p_1 - 3\lambda_0 - 2\delta_0 + \left(p_2 - 3c\lambda_0 - 2c\delta_0\right)e^{-(\lambda_0+\delta_0)\tau} \\ & + e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right)\tau \\ & + e^{-\lambda_0\tau}(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\,\delta_0^{-2}\left(1 - e^{-\delta_0\tau} - \delta_0\tau\, e^{-\delta_0\tau}\right) \equiv \eta_{\lambda_0,\delta_0}\,.\end{aligned}$$

Hence, if we assume that $\eta_{\lambda_0,\delta_0} \neq 0$, then we always have $\gamma_0 \neq 0$. Furthermore, since $\gamma = -\delta_0$ $\left(\delta_0 \neq 0\right)$ is a removable singularity of $G(\gamma)$, we can regard $G(\gamma)$ as a entire function with

$$\begin{aligned}G(-\delta_0) = {} & \delta_0\left(1 + ce^{-\lambda_0\tau}\right) + p_1 - 3\lambda_0 - 2\delta_0 + \left(p_2 - 3c\lambda_0 - 2c\delta_0\right)e^{-\lambda_0\tau} \\ & - e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right)\delta_0^{-1}\left(1 - e^{\delta_0\tau}\right) \\ & - e^{-\lambda_0\tau}(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\,\delta_0^{-1}\left(\delta_0^{-1}\left(1 - e^{-\delta_0\tau}\right) - \tau\right)\end{aligned}$$

or

$$\begin{aligned}\delta_0 G(-\delta_0) = {} & -\delta_0^2 + \delta_0\left(p_1 - 3\lambda_0\right) + \delta_0\left(p_2 - 3c\lambda_0 - 2c\delta_0\right)e^{-\lambda_0\tau} \\ & - e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right) \\ & + e^{-\lambda_0\tau}\left(2c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right) \\ & - e^{-\lambda_0\tau}(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\,\delta_0^{-1}\left(1 - e^{-\delta_0\tau}\right) \\ & + e^{-\lambda_0\tau}(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\tau\ .\end{aligned}$$

By using (1.5), we derive

$$\begin{aligned}\delta_0 G(-\delta_0) = {} & -\left(q_1 + 2p_1\lambda_0 - 3\lambda_0^2\right) - e^{-\lambda_0\tau}\left(q_2 + 2p_2\lambda_0 - 3c\lambda_0^2\right) \\ & + e^{-\lambda_0\tau}(p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3)\tau \equiv \beta_{\lambda_0}\,.\end{aligned}$$

But, by the definition of $\beta_{\lambda_0} \neq 0$, a root of the characteristic equation (1.18) must become $\gamma_0 \neq -\delta_0$. Consequently, it must be $\delta_0 \neq 0$ real root of the characteristic equation (1.5) and $\gamma_0 \neq 0$, $\gamma_0 \neq -\delta_0$ real root of the characteristic equation (1.18).

Define $\mu_{\lambda_0,\delta_0,\gamma_0}$ by (2.3). It is clear $\mu_{\lambda_0,\delta_0,\gamma_0}$ is positive. So, (2.3) can equivalently be written as follows

$$0 < \mu_{\lambda_0,\delta_0,\gamma_0} < 1\,. \tag{2.5}$$

Furthermore, for the real constant $\xi_{\lambda_0,\delta_0,\gamma_0}$ defined by (2.1), we have

$$\begin{aligned}\left|\xi_{\lambda_0,\delta_0,\gamma_0}\right| \leq {} & \left|e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left[\left(p_2 - 3c\lambda_0 - 2c\delta_0 - c\gamma_0\right)\tau + c\right]\right| \\ & + \left|e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right)\int_0^\tau e^{-\gamma_0 s}s\,ds\right|\end{aligned}$$

$$+\left|e^{-\lambda_0\tau}\left(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right)\int_0^{\tau}e^{-\delta_0 s}\left\{\int_0^{s}e^{-\gamma_0 u}u\,du\right\}ds\right|$$

$$\le e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left[\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right|\tau+|c|\right]$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right|\int_0^{\tau}e^{-\gamma_0 s}s\,ds$$

$$+e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right|\int_0^{\tau}e^{-\delta_0 s}\left\{\int_0^{s}e^{-\gamma_0 u}u\,du\right\}ds\equiv\mu_{\lambda_0,\delta_0,\gamma_0}\,.$$

That is,

$$|\,\xi_{\lambda_0,\delta_0,\gamma_0}\,|\le\mu_{\lambda_0,\delta_0,\gamma_0}\,. \qquad (2.6)$$

Thus, if we assume that (2.3) is satisfied, i.e., that (2.5) holds, then (2.6) gives $|\,\xi_{\lambda_0,\delta_0,\gamma_0}\,|<1$. This guarantees, in particular, that

$$1+\xi_{\lambda_0,\delta_0,\gamma_0}>0\,.$$

Proof of Theorem 2.1. Let y be the solution of the IVP (1.1) and (1.2). Define the function z by (1.9). By Proposition 1.1, the fact that y is the solution of the IVP (1.1) and (1.2) is equivalent to the fact that z is the solution of the neutral delay differential equation (1.4) which satisfies the initial condition (1.10). Furthermore, define the function w by (1.22). By Proposition 1.2, the fact that z is the solution of the neutral delay differential equation (1.4) and (1.10) is equivalent to the fact that w is the solution of the neutral delay differential equation (1.17) which satisfies the initial condition (1.23). Set

$$\Omega(t)=e^{-\gamma_0 t}w(t) \quad \text{for all } t\in[-\tau,\infty)\,. \qquad (2.7)$$

Then, using the fact that γ_0 is a real root of the characteristic equation (1.18), from (1.17) we obtain, for every $t\ge 0$,

$$\Omega'(t)+ce^{-(\lambda_0+\delta_0+\gamma_0)\tau}\Omega'(t-\tau)=(p_1-3\lambda_0-2\delta_0-\gamma_0)\Omega(t)$$

$$+e^{-(\lambda_0+\delta_0+\gamma_0)\tau}(p_2-3c\lambda_0-2c\delta_0-c\gamma_0)\Omega(t-\tau)$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2)\right)\int_0^{\tau}e^{-\gamma_0 s}\,\Omega(t-s)ds$$

$$+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau}e^{-\delta_0 s}\left\{\int_0^{s}e^{-\gamma_0 u}\,\Omega(t-u)du\right\}ds\,. \qquad (2.8)$$

Moreover, the initial condition (1.29) can be equivalently written

$$\Omega(t)=e^{-\gamma_0 t}\Phi_2(\lambda_0,\delta_0;\phi)(t) \quad \text{for } t\in[-\tau,0]\,, \qquad (2.9)$$

where $\Phi_2(\lambda_0,\delta_0;\phi)$ is defined by (1.21). Furthermore, using the fact that γ_0 is a real root of (1.18) and taking into account (2.9), we can verify that (2.8) is equivalent to

$$\Omega(t)+ce^{-(\lambda_0+\delta_0+\gamma_0)\tau}\Omega(t-\tau)=(p_1-3\lambda_0-2\delta_0-\gamma_0)\int_0^{t}\Omega(s)ds$$

$$+e^{-(\lambda_0+\delta_0+\gamma_0)\tau}(p_2-3c\lambda_0-2c\delta_0-c\gamma_0)\int_0^{t}\Omega(s-\tau)ds$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left(c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2)\right)\int_0^{\tau}e^{-\gamma_0 s}\left\{\int_0^{t}\Omega(u-s)du\right\}ds$$

$$+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^{\tau}e^{-\delta_0 s}\left\{\int_0^{s}e^{-\gamma_0 u}\left\{\int_0^{t}\Omega(\omega-u)d\omega\right\}du\right\}ds$$

$$+\Omega(0)+ce^{-(\lambda_0+\delta_0+\gamma_0)\tau}\Omega(-\tau)\,,$$

$$\Omega(t)+ce^{-(\lambda_0+\delta_0+\gamma_0)\tau}\Omega(t-\tau)=(p_1-3\lambda_0-2\delta_0-\gamma_0)\int_0^t\Omega(s)ds$$

$$+e^{-(\lambda_0+\delta_0+\gamma_0)\tau}(p_2-3c\lambda_0-2c\delta_0-c\gamma_0)\int_{-\tau}^{t-\tau}\Omega(s)ds$$

$$+e^{-(\lambda_0+\delta_0)\tau}\Big(c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2)\Big)\int_0^\tau e^{-\gamma_0 s}\left\{\int_{-s}^{t-s}\Omega(u)du\right\}ds$$

$$+e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^\tau e^{-\delta_0 s}\left\{\int_0^s e^{-\gamma_0 u}\left\{\int_{-u}^{t-u}\Omega(\omega)d\omega\right\}du\right\}ds$$

$$+\Omega(0)+ce^{-(\lambda_0+\delta_0+\gamma_0)\tau}\Omega(-\tau),$$

$$\Omega(t)+ce^{-(\lambda_0+\delta_0+\gamma_0)\tau}\Omega(t-\tau)=-e^{-(\lambda_0+\delta_0+\gamma_0)\tau}(p_2-3c\lambda_0-2c\delta_0-c\gamma_0)\int_{t-\tau}^{t}\Omega(s)ds$$

$$-e^{-(\lambda_0+\delta_0)\tau}\Big(c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2)\Big)\int_0^\tau e^{-\gamma_0 s}\left\{\int_{t-s}^{t}\Omega(u)du\right\}ds$$

$$-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^\tau e^{-\delta_0 s}\left\{\int_0^s e^{-\gamma_0 u}\left\{\int_{t-u}^{t}\Omega(\omega)d\omega\right\}du\right\}ds+K(\lambda_0,\delta_0,\gamma_0;\phi), \quad (2.10)$$

where $K(\lambda_0,\delta_0,\gamma_0;\phi)$ is defined by (2.2). Next, we define

$$\Theta(t)=\Omega(t)-\frac{K(\lambda_0,\delta_0,\gamma_0;\phi)}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}, \quad \text{for } t\geq-\tau, \quad (2.11)$$

where $\xi_{\lambda_0,\delta_0,\gamma_0}$ is defined by (2.1). Then we can see that (2.10) reduces to the following equivalent equation

$$\Theta(t)+ce^{-(\lambda_0+\delta_0+\gamma_0)\tau}\Theta(t-\tau)=-e^{-(\lambda_0+\delta_0+\gamma_0)\tau}(p_2-3c\lambda_0-2c\delta_0-c\gamma_0)\int_{t-\tau}^{t}\Theta(s)ds$$

$$-e^{-(\lambda_0+\delta_0)\tau}\Big(c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2)\Big)\int_0^\tau e^{-\gamma_0 s}\left\{\int_{t-s}^{t}\Theta(u)du\right\}ds$$

$$-e^{-\lambda_0\tau}(p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3)\int_0^\tau e^{-\delta_0 s}\left\{\int_0^s e^{-\gamma_0 u}\left\{\int_{t-u}^{t}\Theta(\omega)d\omega\right\}du\right\}ds \quad (2.12)$$

for all $t\geq 0$. On the other hand, the initial condition (2.9) can be equivalently written

$$\Theta(t)=e^{-\gamma_0 t}\Phi_2(\lambda_0,\delta_0;\phi)(t)-\frac{K(\lambda_0,\delta_0,\gamma_0;\phi)}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}, \quad \text{for } t\in[-\tau,0]. \quad (2.13)$$

Now, we shall prove that

$$\lim_{t\to\infty}\Theta(t)=0. \quad (2.14)$$

Define

$$M(\lambda_0,\delta_0,\gamma_0;\phi)=\max_{-\tau\leq t\leq 0}\left|e^{-\gamma_0 t}\Phi_2(\lambda_0,\delta_0;\phi)(t)-\frac{K(\lambda_0,\delta_0,\gamma_0;\phi)}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\right| \quad (2.15)$$

It follows from (2.13) and (2.15) that

$$|\Theta(t)|\leq M(\lambda_0,\delta_0,\gamma_0;\phi) \quad \text{for } -\tau\leq t\leq 0. \quad (2.16)$$

We will show that $M(\lambda_0,\delta_0,\gamma_0;\phi)$ is a bound of the function Θ on the whole interval $[-\tau,\infty)$, i.e., that

$$|\Theta(t)|\leq M(\lambda_0,\delta_0,\gamma_0;\phi) \quad \text{for all } t\geq-\tau. \quad (2.17)$$

For this purpose, we consider an arbitrary positive real number ε. We claim that

$$|\Theta(t)|< M(\lambda_0,\delta_0,\gamma_0;\phi)+\varepsilon \quad \text{for every } t\geq-\tau. \quad (2.18)$$

Otherwise, since (2.16) implies that $|\Theta(t)| < M(\lambda_0,\delta_0,\gamma_0;\phi)+\varepsilon$ for $-\tau \le t \le 0$, there exists a point $t^* > 0$ such that

$$|\Theta(t)| < M(\lambda_0,\delta_0,\gamma_0;\phi)+\varepsilon, \quad \text{for } -\tau \le t < t^* \quad \text{and} \quad |\Theta(t^*)| = M(\lambda_0,\delta_0,\gamma_0;\phi)+\varepsilon .$$

Then, by taking into account the definition of $\mu_{\lambda_0,\delta_0,\gamma_0}$ by (2.3) and using (2.5), from (2.12) we obtain

$$\begin{aligned}
&M(\lambda_0,\delta_0,\gamma_0;\phi)+\varepsilon = |\Theta(t^*)| \\
&\le |c|\, e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|\Theta(t^*-\tau)\right| + e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right| \int_{t^*-\tau}^{t^*} |\Theta(s)|\,ds \\
&\qquad + e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right| \int_0^{\tau} e^{-\gamma_0 s}\left\{\int_{t^*-s}^{t^*}|\Theta(u)|\,du\right\}ds \\
&\qquad + e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right| \int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{s} e^{-\gamma_0 u}\left\{\int_{t^*-u}^{t^*}|\Theta(\omega)|\,d\omega\right\}du\right\}ds \\
&\qquad \le \Big\{ e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left[\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right|\tau+|c|\right] \\
&\qquad + e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right| \int_0^{\tau} e^{-\gamma_0 s}\, s\,ds \\
&\qquad + e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right| \int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{s} e^{-\gamma_0 u}\, u\,du\right\}ds\Big\}\left(M(\lambda_0,\delta_0,\gamma_0;\phi)+\varepsilon\right), \\
&\qquad = \mu_{\lambda_0,\delta_0,\gamma_0}\, M(\lambda_0,\delta_0,\gamma_0;\phi) \\
&\qquad < M(\lambda_0,\delta_0,\gamma_0;\phi)+\varepsilon .
\end{aligned}$$

We have thus arrived at a contradiction, which establishes our claim, i.e., that (2.18) holds true. As (2.18) is satisfied for all real numbers $\varepsilon > 0$, it follows that (2.17) is always fulfilled. Furthermore, by using (2.17), from (2.12) we get, for every $t \ge 0$,

$$\begin{aligned}
&|\Theta(t)| \le |c|\, e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|\Theta(t-\tau)\right| + e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right| \int_{t-\tau}^{t} |\Theta(s)|\,ds \\
&\qquad + e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right| \int_0^{\tau} e^{-\gamma_0 s}\left\{\int_{t-s}^{t}|\Theta(u)|\,du\right\}ds \\
&\qquad + e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right| \int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{s} e^{-\gamma_0 u}\left\{\int_{t-u}^{t}|\Theta(\omega)|\,d\omega\right\}du\right\}ds \\
&\qquad \le \Big\{ e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left[\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right|\tau+|c|\right] \\
&\qquad + e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right| \int_0^{\tau} e^{-\gamma_0 s}\, s\,ds \\
&\qquad + e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right| \int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{s} e^{-\gamma_0 u}\, u\,du\right\}ds\Big\}\, M(\lambda_0,\delta_0,\gamma_0;\phi) .
\end{aligned}$$

Thus, by taking into account the definition of $\mu_{\lambda_0,\delta_0,\gamma_0}$ by (2.3), we have

$$|\Theta(t)| \le \mu_{\lambda_0,\delta_0,\gamma_0}\, M(\lambda_0,\delta_0,\gamma_0;\phi) \quad \text{for every} \quad t \ge 0 . \tag{2.19}$$

By using (2.12) and taking into account the definition of $\mu_{\lambda_0,\delta_0,\gamma_0}$ by (2.3) as well as taking into account (2.17) and (2.19), one can prove, by an easy induction, that the function Θ satisfies

$$|\Theta(t)| \le \left(\mu_{\lambda_0,\delta_0,\gamma_0}\right)^n M(\lambda_0,\delta_0,\gamma_0;\phi), \quad \text{for all} \quad t \ge n\tau-\tau , \quad (n = 0,1,\ldots) . \tag{2.20}$$

Because of (2.5), we have

$$\lim_{n\to\infty}[\mu_{\lambda_0,\delta_0,\gamma_0}]^n = 0 . \qquad (2.21)$$

In view of (2.21), it follows from (2.20) that $\lim_{t\to\infty}\Theta(t) = 0$, i.e., (2.14) holds true.

Finally, by (1.22), (2.7) and (2.11), we have

$$\Theta(t) = e^{-\gamma_0 t}\left[e^{-(\lambda_0+\delta_0)t} y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} - \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}} \right] - \frac{K(\lambda_0,\delta_0,\gamma_0;\phi)}{1+\xi_{\lambda_0,\delta_0,\gamma_0}} , \qquad (2.22)$$

for $t \geq -\tau$. In view of this equality, (2.4) coincides with (2.14). So, the solution y of the IVP (1.1) and (1.2) satisfies (2.4). The proof of the theorem is complete.

3. An Estimate of the Solutions and a Stability Criterion

Our results in this section are Theorem 3.1 below and its corollary.

Theorem3.1. *Let* λ_0 *be a real root of the characteristic equation* (1.3), *and suppose that* $\beta_{\lambda_0} \neq 0$, *where* β_{λ_0} *is defined by* (1.6). *Let* δ_0 *be a real root of the characteristic equation* (1.5), *suppose that* $\eta_{\lambda_0,\delta_0} \neq 0$, *where* $\eta_{\lambda_0,\delta_0}$ *is defined by* (1.19). *Furthermore, let* γ_0 *be a real root of the characteristic equation* (1.18), *and let* $\xi_{\lambda_0,\delta_0,\gamma_0}$ *be defined by* (2.1). *(Note that, because of* $\beta_{\lambda_0} \neq 0$, *we always have* $\delta_0 \neq 0$ *and* $\gamma_0 \neq -\delta_0$. *Furthermore, because of* $\eta_{\lambda_0,\delta_0} \neq 0$, *we always have* $\gamma_0 \neq 0$. *) Set*

$$m(\lambda_0,\delta_0,\gamma_0) = \max\left\{1 \ ,\ e^{\lambda_0\tau} \ ,\ e^{\delta_0\tau} \ ,\ e^{\gamma_0\tau} \ ,\ e^{(\lambda_0+\delta_0)\tau} \ ,\ e^{(\lambda_0+\delta_0+\gamma_0)\tau} \right\}. \qquad (3.1)$$

Assume that $\mu_{\lambda_0,\delta_0,\gamma_0} < 1$ *holds, where* $\mu_{\lambda_0,\delta_0,\gamma_0}$ *is defined by* (2.3). *(This assumption guarantees that* $1+\xi_{\lambda_0,\delta_0,\gamma_0} > 0$. *) Then the solution* y *of the IVP* (1.1) *and* (1.2) *satisfies*

$$|\, y(t)| \leq \left\{ \frac{k_{\lambda_0}}{|\,\beta_{\lambda_0}\,|} e^{\lambda_0 t} + \frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|} e^{(\lambda_0+\delta_0)t} + g(\lambda_0,\delta_{0,}\gamma_0) e^{(\lambda_0+\delta_0+\gamma_0)t} \right\} \|\phi\|_{C^2} , \qquad (3.2)$$

for all $t \geq 0$, *where*

$$\begin{aligned} k_{\lambda_0} &= 1 + |\,c\,| + |\,\lambda_0 - p_1\,| + |\,\lambda_0 c - p_2\,| + |\,\lambda_0^2 - p_1\lambda_0 - q_1\,| + |\,c\lambda_0^2 - p_2\lambda_0 - q_2\,| \\ &\quad + e^{-\lambda_0\tau}\left| p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3 \right| \int_{-\tau}^{0} e^{-\lambda_0 s} ds \ , \end{aligned} \qquad (3.3)$$

$$\begin{aligned} h_{\lambda_0,\delta_0} &= m(\lambda_0,\delta_0,\gamma_0)\left(1 + \frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\right)\Bigg\{ \left(1 + |\lambda_0| + |\delta_0|\ \right)\left(1 + |c| e^{-\lambda_0\tau}\right) \\ &\quad + \left|p_1 - 3\lambda_0 - 2\delta_0\right| + e^{-\lambda_0\tau}\left|p_2 - 3c\lambda_0 - 2c\delta_0\right| \\ &\quad + e^{-(\lambda_0+\delta_0)\tau}\left|(p_2 - 3c\lambda_0)\delta_0 + q_2 + 2p_2\lambda_0 - 3c\lambda_0^2 - c\delta_0^2\right| \int_{-\tau}^{0} e^{-\delta_0 s} ds \Bigg]\Bigg\} \|\phi\|_{C^2} , \end{aligned} \qquad (3.4)$$

$$g(\lambda_0,\delta_{0,}\gamma_0) = \mu_{\lambda_0,\delta_0,\gamma_0} m^2(\lambda_0,\delta_0,\gamma_0)\left(1 + \frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|} + \frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|}\right) + \frac{1+\mu_{\lambda_0,\delta_0,\gamma_0}}{1+\xi_{\lambda_0,\delta_0,\gamma_0}} \ell_{\lambda_0,\delta_0,\gamma_0} \ , \qquad (3.5)$$

$$\begin{aligned} \ell_{\lambda_0,\delta_{0,}\gamma_0} &= \Bigg\{ 1 + |c| e^{-(\lambda_0+\delta_0)\tau} + e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|p_2 - 3c\lambda_0 - 2c\delta_0 - c\gamma_0\right| \int_{-\tau}^{0} e^{-\gamma_0 s} ds \\ &\quad + e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2 - p_2\delta_0 + 3c\lambda_0\delta_0 - q_2 - 2p_2\lambda_0 + 3c\lambda_0^2\right| \int_0^{\tau} e^{-\gamma_0 s} \left\{ \int_{-s}^{0} e^{-\gamma_0 u} du \right\} ds \\ &\quad + e^{-\lambda_0\tau}\left|p_2\lambda_0^2 + q_2\lambda_0 + v_2 - c\lambda_0^3\right| \int_0^{\tau} e^{-\delta_0 s}\left\{ \int_0^{s} e^{-\gamma_0 u} \left\{ \int_{-u}^{0} e^{-\gamma_0 \omega} d\omega \right\} du \right\} ds \Bigg\} \end{aligned}$$

$$\times m(\lambda_0,\delta_0,\gamma_0)\left\{1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}+\frac{h_{\lambda_0,\delta_0}}{|\eta_{\lambda_0,\delta_0}|}\right\}, \tag{3.6}$$

The constant $g(\lambda_0,\delta_{0,}\gamma_0)$ *is greater than 1.*

Corollary 3.2. *Let* λ_0, δ_0 *and* γ_0 *be real roots of the characteristic equations* (1.3), (1.5) *and* (1.18), *respectively. Suppose that* $\beta_{\lambda_0}\neq 0$ *and* $\eta_{\lambda_0,\delta_0}\neq 0$, *where* β_{λ_0} *and* $\eta_{\lambda_0,\delta_0}$ *are defined by* (1.6) *and* (1.19), *respectively. (Note that, because of* $\beta_{\lambda_0}\neq 0$, *we always have* $\delta_0\neq 0$ *and* $\gamma_0\neq-\delta_0$. *Furthermore, because of* $\eta_{\lambda_0,\delta_0}\neq 0$, *we always have* $\gamma_0\neq 0$ *.)*

Assume that $\mu_{\lambda_0,\delta_0,\gamma_0}<1$ *holds, where* $\mu_{\lambda_0,\delta_0,\gamma_0}$ *is defined by* (2.3). *Then the trivial solution of the neutral delay differential equation (1.1) is stable if* $\lambda_0\leq 0$, $\lambda_0+\delta_0\leq 0, \lambda_0+\delta_0+\gamma_0\leq 0$ *and it is asymptotically stable if* $\lambda_0<0$, $\lambda_0+\delta_0<0$, $\lambda_0+\delta_0+\gamma_0<0$.

Proof of Theorem 3.1. First of all, we observe that, for any real number a, it holds $\max\limits_{-\tau\leq t\leq 0} e^{-at}=\max\{1\ ,\ e^{a\tau}\}$. So, using (3.1), we immediately see that

$$e^{-\lambda_0 t}\ ,\ e^{-\delta_0 t}\ ,\ e^{-\gamma_0 t}\ ,\ e^{-(\lambda_0+\delta_0)t}\ ,\ e^{-(\lambda_0+\delta_0+\gamma_0)t}\leq m(\lambda_0,\delta_0,\gamma_0), \tag{3.7}$$

for $-\tau\leq t\leq 0$. These inequalities will be frequently used later.

Define $L(\lambda_0;\phi)$ by (1.7). Then

$$\begin{aligned}|L(\lambda_0;\phi)|\leq&|\phi''(0)|+|c\|\phi''(-\tau)|+|\lambda_0-p_1\|\phi'(0)|+|\lambda_0 c-p_2\|\phi'(-\tau)|\\&+|\lambda_0^2-p_1\lambda_0-q_1\|\phi(0)|+|c\lambda_0^2-p_2\lambda_0-q\|\phi(-\tau)|\\&+e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right|\int_{-\tau}^{0}e^{-\lambda_0 s}\,|\phi(s)|\,ds\\\leq&\Big\{(1+|c|)\|\phi''\|+(|\lambda_0-p_1|+|\lambda_0 c-p_2|)\|\phi'\|+\Big(|\lambda_0^2-p_1\lambda_0-q_1|\\&+|c\lambda_0^2-p_2\lambda_0-q_2|+e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right|\int_{-\tau}^{0}e^{-\lambda_0 s}ds\Big)\|\phi\|_{C^2}.\end{aligned}$$

By (3.3), we have

$$|L(\lambda_0;\phi)|\leq k_{\lambda_0}\|\phi\|_{C^2}. \tag{3.8}$$

Consider the function $\Phi_1(\lambda_0;\phi)$ defined by (1.8). Then, by (3.1), we have

$$\|\Phi_1(\lambda_0;\phi)\|\leq m(\lambda_0,\delta_0,\gamma_0)\|\phi\|+\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}$$

and so, in view of (3.8),

$$\|\Phi_1(\lambda_0;\phi)\|\leq m(\lambda_0,\delta_0,\gamma_0)\|\phi\|+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\|\phi\|_{C^2}.$$

Therefore,

$$\|\Phi_1(\lambda_0;\phi)\|\leq m(\lambda_0,\delta_0,\gamma_0)\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right)\|\phi\|_{C^2}. \tag{3.9}$$

Furthermore,

$$\left\|(\Phi_1(\lambda_0;\phi))'\right\|\leq m(\lambda_0,\delta_0,\gamma_0)(\|\phi'\|+|\lambda_0|\|\phi\|)$$

where $(\Phi_1(\lambda_0;\phi))'$ is derivative of $\Phi_1(\lambda_0;\phi)$, andtherefore,

$$\left\|(\Phi_1(\lambda_0;\phi))'\right\|\leq m(\lambda_0,\delta_0,\gamma_0)(1+|\lambda_0|)\|\phi\|_{C^2}. \tag{3.10}$$

Let us consider the constant $R(\lambda_0,\delta_0;\phi)$ defined by (1.20). Then, by (3.9) and (3.10), we have

$$\begin{aligned}
\left|R(\lambda_0,\delta_0;\phi)\right| \le{}& \left|\left(\Phi_1(\lambda_0;\phi)\right)'(0)\right| + \left|\delta_0\right|\left|\Phi_1(\lambda_0;\phi)(0)\right| \\
&+\left|c\right|e^{-\lambda_0\tau}\left(\left|\left(\Phi_1(\lambda_0;\phi)\right)'(-\tau)\right| + \left|\delta_0\right|\left|\Phi_1(\lambda_0;\phi)(-\tau)\right|\right) + \left|p_1-3\lambda_0-2\delta_0\right|\left|\Phi_1(\lambda_0;\phi)(0)\right| \\
&+e^{-\lambda_0\tau}\left|p_2-3c\lambda_0-2c\delta_0\right|\left|\Phi_1(\lambda_0;\phi)(-\tau)\right| \\
&+e^{-(\lambda_0+\delta_0)\tau}\left|(p_2-3c\lambda_0)\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2\right|\int_{-\tau}^{0}e^{-\delta_0 s}\left|\Phi_1(\lambda_0;\phi)(s)\right|ds \\
\le{}& \left\|\left(\Phi_1(\lambda_0;\phi)\right)'\right\| + \left|\delta_0\right|\left\|\Phi_1(\lambda_0;\phi)\right\| + \left|c\right|e^{-\lambda_0\tau}\left(\left\|\left(\Phi_1(\lambda_0;\phi)\right)'\right\| + \left|\delta_0\right|\left\|\Phi_1(\lambda_0;\phi)\right\|\right) \\
&+\left|p_1-3\lambda_0-2\delta_0\right|\left\|\Phi_1(\lambda_0;\phi)\right\| + e^{-\lambda_0\tau}\left|p_2-3c\lambda_0-2c\delta_0\right|\left\|\Phi_1(\lambda_0;\phi)\right\| \\
&+e^{-(\lambda_0+\delta_0)\tau}\left|(p_2-3c\lambda_0)\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2\right|\int_{-\tau}^{0}e^{-\delta_0 s}\left\|\Phi_1(\lambda_0;\phi)\right\|ds \\
\le{}& m(\lambda_0,\delta_0,\gamma_0)\left\{1+\left|\lambda_0\right|+\left|\delta_0\right|\left(1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\right)+\left|c\right|e^{-\lambda_0\tau}\left(1+\left|\lambda_0\right|+\left|\delta_0\right|\left(1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\right)\right)\right. \\
&+\left|p_1-3\lambda_0-2\delta_0\right|\left(1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\right)+e^{-\lambda_0\tau}\left|p_2-3c\lambda_0-2c\delta_0\right|\left(1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\right) \\
&\left.\left.+e^{-(\lambda_0+\delta_0)\tau}\left|(p_2-3c\lambda_0)\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2\right|\int_{-\tau}^{0}e^{-\delta_0 s}\left(1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\right)ds\right]\right\}\left\|\phi\right\|_{C^2} \\
\le{}& m(\lambda_0,\delta_0,\gamma_0)\left(1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\right)\left\{\left(1+\left|\lambda_0\right|+\left|\delta_0\right|\right)\left(1+\left|c\right|e^{-\lambda_0\tau}\right)\right. \\
&+\left|p_1-3\lambda_0-2\delta_0\right|+e^{-\lambda_0\tau}\left|p_2-3c\lambda_0-2c\delta_0\right| \\
&\left.\left.+e^{-(\lambda_0+\delta_0)\tau}\left|(p_2-3c\lambda_0)\delta_0+q_2+2p_2\lambda_0-3c\lambda_0^2-c\delta_0^2\right|\int_{-\tau}^{0}e^{-\delta_0 s}ds\right]\right\}\left\|\phi\right\|_{C^2}.
\end{aligned}$$

By (3.4), we have

$$|R(\lambda_0,\delta_0;\phi)| \le h_{\lambda_0,\delta_0}\left\|\phi\right\|_{C^2}.$$

(3.11)

Consider the function $\Phi_2(\lambda_0,\delta_0;\phi)$ defined by (1.21). Then, by (3.1), we have

$$\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\| \le m(\lambda_0,\delta_0,\gamma_0)\left\|\phi\right\| + \frac{\left|L(\lambda_0;\phi)\right|}{\left|\beta_{\lambda_0}\right|}m(\lambda_0,\delta_0,\gamma_0) + \frac{\left|R(\lambda_0,\delta_0;\phi)\right|}{\left|\eta_{\lambda_0,\delta_0}\right|}$$

and so, in view of (3.8) and (3.11),

$$\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\| \le m(\lambda_0,\delta_0,\gamma_0)\left\|\phi\right\| + m(\lambda_0,\delta_0,\gamma_0)\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}\left\|\phi\right\|_{C^2} + \frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|}\left\|\phi\right\|_{C^2}.$$

Therefore,

$$\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\| \le m(\lambda_0,\delta_0,\gamma_0)\left\{1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}+\frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|}\right\}\left\|\phi\right\|_{C^2}. \tag{3.12}$$

Let us consider the constant $K(\lambda_0,\delta_0,\gamma_0;\phi)$ defined by (2.2). Then, by (3.12), we have

$$\left|K(\lambda_0,\delta_0,\gamma_0;\phi)\right| \le \left|\Phi_2(\lambda_0,\delta_0;\phi)(0)\right| + \left|c\right|e^{-(\lambda_0+\delta_0)\tau}\left|\Phi_2(\lambda_0,\delta_0;\phi)(-\tau)\right|$$

$$+e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right|\int_{-\tau}^{0}e^{-\gamma_0 s}\left|\Phi_2(\lambda_0,\delta_0;\phi)(s)\right|ds$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right|\int_0^{\tau}e^{-\gamma_0 s}\left\{\int_{-s}^{0}e^{-\gamma_0 u}\left|\Phi_2(\lambda_0,\delta_0;\phi)(u)\right|du\right\}ds$$

$$+e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right|\int_0^{\tau}e^{-\delta_0 s}\left\{\int_0^{s}e^{-\gamma_0 u}\left\{\int_{-u}^{0}e^{-\gamma_0\omega}\left|\Phi_2(\lambda_0,\delta_0;\phi)(\omega)\right|d\omega\right\}du\right\}ds$$

$$\leq\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\|+|c|e^{-(\lambda_0+\delta_0)\tau}\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\|$$

$$+e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right|\int_{-\tau}^{0}e^{-\gamma_0 s}\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\|ds$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right|\int_0^{\tau}e^{-\gamma_0 s}\left\{\int_{-s}^{0}e^{-\gamma_0 u}\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\|du\right\}ds$$

$$+e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right|\int_0^{\tau}e^{-\delta_0 s}\left\{\int_0^{s}e^{-\gamma_0 u}\left\{\int_{-u}^{0}e^{-\gamma_0\omega}\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\|d\omega\right\}du\right\}ds$$

$$\leq\left\{1+|c|e^{-(\lambda_0+\delta_0)\tau}+e^{-(\lambda_0+\delta_0+\gamma_0)\tau}\left|p_2-3c\lambda_0-2c\delta_0-c\gamma_0\right|\int_{-\tau}^{0}e^{-\gamma_0 s}ds\right.$$

$$+e^{-(\lambda_0+\delta_0)\tau}\left|c\delta_0^2-p_2\delta_0+3c\lambda_0\delta_0-q_2-2p_2\lambda_0+3c\lambda_0^2\right|\int_0^{\tau}e^{-\gamma_0 s}\left\{\int_{-s}^{0}e^{-\gamma_0 u}du\right\}ds$$

$$\left.+e^{-\lambda_0\tau}\left|p_2\lambda_0^2+q_2\lambda_0+v_2-c\lambda_0^3\right|\int_0^{\tau}e^{-\delta_0 s}\left\{\int_0^{s}e^{-\gamma_0 u}\left\{\int_{-u}^{0}e^{-\gamma_0\omega}d\omega\right\}du\right\}ds\right\}$$

$$\times m(\lambda_0,\delta_0,\gamma_0)\left\{1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}+\frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|}\right\}\|\phi\|_{C^2}.$$

By (3.6), we have

$$|K(\lambda_0,\delta_0,\gamma_0;\phi)|\leq\ell_{\lambda_0,\delta_0,\gamma_0}\|\phi\|_{C^2}.$$

(3.13)

Let $\xi_{\lambda_0,\delta_0,\gamma_0}$ be defined by (2.1), (Note that, because of $\mu_{\lambda_0,\delta_0,\gamma_0}<1$ by (2.3), we always have $1+\xi_{\lambda_0,\delta_0,\gamma_0}>0$.) and define $M(\lambda_0,\delta_0,\gamma_0;\phi)$ by (2.15). Then, by using (3.1), we have

$$M(\lambda_0,\delta_0,\gamma_0;\phi)\leq m(\lambda_0,\delta_0,\gamma_0)\left\|\Phi_2(\lambda_0,\delta_0;\phi)\right\|+\frac{\left|K(\lambda_0,\delta_0,\gamma_0;\phi)\right|}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}.$$

So, by virtue of (3.12) and (3.13), we have

$$M(\lambda_0,\delta_0,\gamma_0;\phi)\leq\left\{m^2(\lambda_0,\delta_0,\gamma_0)\left(1+\frac{k_{\lambda_0}}{\left|\beta_{\lambda_0}\right|}+\frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|}\right)+\frac{\ell_{\lambda_0,\delta_0,\gamma_0}}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\right\}\|\phi\|_{C^2}. \tag{3.14}$$

Now, let y be the solution of the IVP (1.1) and (1.2), and define the functions z by (1.9) and w by (1.22). Also, we define the functions Ω and Θ by (2.7) and (2.11), respectively. Note that (2.1) (which is a consequence of the assumption (2.3)) states that $1+\xi_{\lambda_0,\delta_0,\gamma_0}>0$. Then, as in the proof of the Theorem 2.1, we show that (2.19) and (2.22) are satisfied. We shall prove that y satisfies (3.2), where the constants k_{λ_0}, h_{λ_0,δ_0}, $g(\lambda_0,\delta_{0,}\gamma_0)$ are $\ell_{\lambda_0,\delta_{0,}\gamma_0}$ are defined by (3.3), (3.4), (3.5) and (3.6), respectively.

From (2.22) it follows that

$$y(t)=\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{\lambda_0 t}+\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}e^{(\lambda_0+\delta_0)t}+\left[\Theta(t)+\frac{K(\lambda_0,\delta_0,\gamma_0;\phi)}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\right]e^{(\lambda_0+\delta_0+\gamma_0)t}$$

for $t\geq 0$ and consequently

$$|y(t)|\leq\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}e^{\lambda_0 t}+\frac{|R(\lambda_0,\delta_0;\phi)|}{|\eta_{\lambda_0,\delta_0}|}e^{(\lambda_0+\delta_0)t}+\left[|\Theta(t)|+\frac{|K(\lambda_0,\delta_0,\gamma_0;\phi)|}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\right]e^{(\lambda_0+\delta_0+\gamma_0)t}\ .$$

Thus, using (2.19), we obtain

$$|y(t)|\leq\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}e^{\lambda_0 t}+\frac{|R(\lambda_0,\delta_0;\phi)|}{|\eta_{\lambda_0,\delta_0}|}e^{(\lambda_0+\delta_0)t}$$
$$+\left[\mu_{\lambda_0,\delta_0,\gamma_0}M(\lambda_0,\delta_0,\gamma_0;\phi)+\frac{|K(\lambda_0,\delta_0,\gamma_0;\phi)|}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\right]e^{(\lambda_0+\delta_0+\gamma_0)t} \qquad (3.15)$$

for $t\geq 0$. Using (3.8) and (3.11), we obtain

$$\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\leq\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\|\phi\|_{C^2} \qquad (3.16)$$

and

$$\frac{|R(\lambda_0,\delta_0;\phi)|}{|\eta_{\lambda_0,\delta_0}|}\leq\frac{h_{\lambda_0,\delta_0}}{|\eta_{\lambda_0,\delta_0}|}\|\phi\|_{C^2}\ . \qquad (3.17)$$

Moreover, by the use of (3.13) and (3.14), we get

$$\mu_{\lambda_0,\delta_0,\gamma_0}M(\lambda_0,\delta_0,\gamma_0;\phi)+\frac{|K(\lambda_0,\delta_0,\gamma_0;\phi)|}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}$$
$$\leq\left\{\mu_{\lambda_0,\delta_0,\gamma_0}\left\{m^2(\lambda_0,\delta_0,\gamma_0)\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}+\frac{h_{\lambda_0,\delta_0}}{|\eta_{\lambda_0,\delta_0}|}\right)+\frac{\ell_{\lambda_0,\delta_0,\gamma_0}}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\right\}+\frac{\ell_{\lambda_0,\delta_0,\gamma_0}}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\right\}\|\phi\|_{C^2}$$
$$=\left\{\mu_{\lambda_0,\delta_0,\gamma_0}m^2(\lambda_0,\delta_0,\gamma_0)\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}+\frac{h_{\lambda_0,\delta_0}}{|\eta_{\lambda_0,\delta_0}|}\right)+\frac{1+\mu_{\lambda_0,\delta_0,\gamma_0}}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\ell_{\lambda_0,\delta_0,\gamma_0}\right\}\|\phi\|_{C^2}\ .$$

So, because of (3.5), we have

$$\mu_{\lambda_0,\delta_0,\gamma_0}M(\lambda_0,\delta_0,\gamma_0;\phi)+\frac{|K(\lambda_0,\delta_0,\gamma_0;\phi)|}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\leq g(\lambda_0,\delta_{0,}\gamma_0)\|\phi\|_{C^2}\ . \qquad (3.18)$$

Using (3.16), (3.17) and (3.18), we immediately see that (3.15) implies (3.2). Hence, (3.2) has been proved.

Finally, we will establish that the constant $g(\lambda_0,\delta_{0,}\gamma_0)$ is greater than 1. By (2.6) and since $1+\xi_{\lambda_0,\delta_0,\gamma_0}>0$, we have

$$0<1+\xi_{\lambda_0,\delta_0,\gamma_0}\leq 1+|\xi_{\lambda_0,\delta_0,\gamma_0}|\leq 1+\mu_{\lambda_0,\delta_0,\gamma_0}\ ,$$

which ensures that

$$1\leq\frac{1+\mu_{\lambda_0,\delta_0,\gamma_0}}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\ .$$

Furthermore, as $m(\lambda_0,\delta_0,\gamma_0)\geq 1$, from (3.6) we have

$$\ell_{\lambda_0,\delta_0,\gamma_0}>1.$$

Thus, we obtain

$$1<\frac{1+\mu_{\lambda_0,\delta_0,\gamma_0}}{1+\xi_{\lambda_0,\delta_0,\gamma_0}}\ell_{\lambda_0,\delta_0,\gamma_0}\ .$$

Hence, it follows from the definition of $g(\lambda_0,\delta_{0,}\gamma_0)$ by (3.5) that $g(\lambda_0,\delta_{0,}\gamma_0)$ is always greater than 1. The proof of the theorem is now complete.

Proof of Corollary 3.2. Let y be the solution of the IVP (1.1) and (1.2). By Theorem 3.1, the solution y satisfies (3.2), where β_{λ_0}, $\eta_{\lambda_0,\delta_0}$, k_{λ_0}, h_{λ_0,δ_0} and $g(\lambda_0,\delta_{0,}\gamma_0)$ are defined by (1.6), (1.19), (3.3), (3.4) and (3.5), respectively. The constant $g(\lambda_0,\delta_{0,}\gamma_0)$ is greater than 1.

Assume first that $\lambda_0 \le 0$, $\lambda_0+\delta_0 \le 0$ and $\lambda_0+\delta_0+\gamma_0 \le 0$. Then (3.2), gives

$$|\,y(t)\,| \le \left\{ \frac{k_{\lambda_0}}{|\,\beta_{\lambda_0}\,|} + \frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|} + g(\lambda_0,\delta_{0,}\gamma_0) \right\} \|\phi\|_{C^2} \qquad \text{for } t \ge 0 .$$

So, if we set

$$S(\lambda_0,\delta_{0,}\gamma_0) = \frac{k_{\lambda_0}}{|\,\beta_{\lambda_0}\,|} + \frac{h_{\lambda_0,\delta_0}}{\left|\eta_{\lambda_0,\delta_0}\right|} + g(\lambda_0,\delta_{0,}\gamma_0) ,$$

then we have

$$|\,y(t)\,| \le S(\lambda_0,\delta_{0,}\gamma_0) \|\phi\|_{C^2} \qquad \text{for every } t \ge 0 .$$

Since $g(\lambda_0,\delta_{0,}\gamma_0) > 1$, we always have $S(\lambda_0,\delta_{0,}\gamma_0) > 1$. Thus, we obtain

$$|\,y(t)\,| \le S(\lambda_0,\delta_{0,}\gamma_0) \|\phi\|_{C^2} \qquad \text{for all } t \ge -\tau .$$

Using this inequality, we can immediately verify that the trivial solution of (1.1) is stable (at 0).

Next, let us suppose that $\lambda_0 < 0$, $\lambda_0+\delta_0 < 0$ and $\lambda_0+\delta_0+\gamma_0 < 0$. Then the trivial solution of (1.1) is stable (at 0). Furthermore, we see that it follows from (3.2) that the solution y satisfies

$$\lim_{t\to\infty} y(t) = 0 .$$

Hence, the trivial solution of (1.1) is asymptotically stable (at 0).

The proof of Corollary 3.2 is completed.

4. Examples

Example 4.1. Consider

$$\begin{aligned} y'''(t) + \frac{1}{e^4} y'''(t-1) &= \left(\frac{3}{e} - \frac{2}{e^2} - 6 \right) y''(t) - \frac{6}{e^4} y''(t-1) + \left(\frac{9}{e} - \frac{6}{e^2} - 11 \right) y'(t) \\ &\quad - \frac{11}{e^4} y'(t-1) + \left(\frac{6}{e} - \frac{4}{e^2} - 6 \right) y(t) - \frac{6}{e^4} y(t-1) , \qquad t \ge 0 , \\ y(t) &= \phi(t) , \qquad -1 \le t \le 0 , \end{aligned} \tag{4.1}$$

where $\phi(t)$ is an arbitrary twice continuously differentiable initial function on the interval $[-1,0]$. In this example we apply the characteristic equations (1.3), (1.5) and (1.18). That is, the characteristic equation (1.3) is

$$\begin{aligned} \lambda^3 + \lambda^3 \frac{1}{e^4} e^{-\lambda} &= \lambda^2 \left(\frac{3}{e} - \frac{2}{e^2} - 6 \right) + \lambda \left(\frac{9}{e} - \frac{6}{e^2} - 11 \right) + \left(\frac{6}{e} - \frac{4}{e^2} - 6 \right) \\ &\quad + e^{-\lambda} \left(-\lambda^2 \frac{6}{e^4} - \lambda \frac{11}{e^4} - \frac{6}{e^4} \right) , \end{aligned} \tag{4.2}$$

and we see that $\lambda = -1$ is a real root of (4.2). Then, for $\lambda_0 = -1$ the characteristic equation (1.5) is

$$\delta^2 + \delta^2 \frac{1}{e^4} e^{1-\delta} = \left(\frac{3}{e} - \frac{2}{e^2} - 3 \right) \delta - \frac{3}{e^4} \delta e^{1-\delta} + \left(\frac{3}{e} - \frac{2}{e^2} - 2 \right) - \frac{2}{e^4} e^{1-\delta} . \tag{4.3}$$

Therefore, $\delta = \delta_0 = -1$ is a real root of (4.3). Then, for $\lambda_0 = \delta_0 = -1$ the characteristic equation (1.18) is

$$\gamma \left(1 + \frac{1}{e^4} e^{2-\gamma} \right) = \left(\frac{3}{e} - \frac{2}{e^2} - 1 \right) - \left(\frac{1}{e^4} \right) e^{2-\gamma} , \tag{4.4}$$

and we see that $\gamma = \gamma_0 \cong -0.84253$ is a real root of (4.4) andthe conditions of Corollary3.2 are satisfied. That is,

$$\delta_0 = -1 \ne 0 , \; \gamma_0 \cong -0.84253 \ne 0 , \; \gamma_0 \ne -\delta_0$$

and

$$\mu_{\lambda_0,\delta_0,\gamma_0} = \mu_{-1,-1,-0.84253} = \frac{1.1575}{e^4} < 1 .$$

Since $\lambda_0 = -1 < 0, \lambda_0 + \delta_0 = -2 < 0$ and $\lambda_0 + \delta_0 + \gamma_0 = -2.84253 < 0$ the zero solution of (4.1) is asymptotically stable.

Example 4.2. Consider

$$\begin{aligned} y'''(t) + \frac{1}{10} y'''(t-0.5) &= \left(\frac{9}{40} e^2 - \frac{9}{40} e - 6 \right) y''(t) - \frac{6}{10} y''(t-0.5) \\ &\quad + \left(\frac{9}{20} e^2 - \frac{9}{20} e - 8 \right) y'(t) - \frac{8}{10} y'(t-0.5) \ , \quad t \geq 0 , \qquad (4.5) \\ y(t) &= \phi(t) , \quad -0.5 \leq t \leq 0 , \end{aligned}$$

where $\phi(t)$ is an arbitrary twice continuously differentiable initial function on the interval $[-1,0]$. In this example we apply the characteristic equations (1.3), (1.5) and (1.18). That is, the characteristic equation (1.3) is

$$\lambda^3 \left(1 + \frac{1}{10} e^{-\frac{\lambda}{2}} \right) = \lambda^2 \left(\frac{9}{40} e^2 - \frac{9}{40} e - 6 \right) + \lambda \left(\frac{9}{20} e^2 - \frac{9}{20} e - 8 \right) - e^{-\frac{\lambda}{2}} \left(\frac{6}{10} \lambda^2 + \frac{8}{10} \lambda \right), \qquad (4.6)$$

and we see easily that $\lambda = 0$ is a real root of (4.6). Taking $\lambda_0 = 0$, the characteristic equation (1.5) is

$$\delta^2 \left(1 + \frac{1}{10} e^{-\frac{\delta}{2}} \right) = \left(\frac{9}{40} e^2 - \frac{9}{40} e - 6 \right) \delta - \frac{6}{10} \delta e^{-\frac{\delta}{2}} + \left(\frac{9}{20} e^2 - \frac{9}{20} e - 8 \right) - \frac{8}{10} e^{-\frac{\delta}{2}} . \qquad (4.7)$$

Therefore, we see that $\delta = \delta_0 = -2$ is a real root of (4.7). Then, for $\lambda_0 = 0$ and $\delta_0 = -2$ the characteristic equation (1.18) is

$$\gamma \left(1 + \frac{1}{10} e^{1-\frac{\gamma}{2}} \right) = \left(\frac{9}{40} e^2 - \frac{9}{40} e - 6 \right) + 4 - \frac{2}{10} e^{1-\frac{\gamma}{2}} , \qquad (4.8)$$

and we find that $\gamma = \gamma_0 \cong -1.3101$ is a real root of (4.8). Corresponding to the roots $\lambda_0 = 0$, $\delta_0 = -2$ and $\gamma_0 = -1.3101$, the conditions of Corollary 3.2 are satisfied. That is,

$$\delta_0 = -2 \neq 0 \ , \ \gamma_0 \cong -1.3101 \neq 0 \ \gamma_0 \neq -\delta_0$$

and

$$\mu_{\lambda_0,\delta_0,\gamma_0} = \mu_{0,-2,-1.3101} = 0.1345 e^2 \cong 0.99383 < 1.$$

Since $\lambda_0 = 0, \lambda_0 + \delta_0 = -2 < 0$ and $\lambda_0 + \delta_0 + \gamma_0 = -3.3101 < 0$ the zero solution of (4.5) is stable.

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