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Two-Step Block Hybrid Multistep Method for the Solution of $y'' = f(x, y, y')$

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Abstract: This paper proposes a self-starting block hybrid method of order $(6,6,6,6,6,6,6,6,6,6,6,6)^T$ for t solution of general second order ODEs of the form $y'' = f(x, y, y')$, $a \le x \le b$ with associated initial boundary conditions. Derivation of the continuous hybrid formulation was based on the use of hermite polynomial basis function. The continuous hybrid formulation enables us to evaluate and then differentiate at some grid and o grid points to obtained discrete schemes which were used in block form. This approach eliminates the need starting values. The computational burden and computer time wastage involved in the usual reduction of the seco order problem into system of first order equations are avoided by this procedure. The stability properties of c method reveal that the method is consistent and zero stable, hence convergent. Numerical results suggest that t method performs favorably with the existing methods.

Keywords: Block method; Hermite polynomial; Interpolation and collocation; Basis function; Second order ODEs.

1. Introduction

The desire to solve physical problems often leads to mathematical models involving an equation in which a function and its derivative play important roles. Equations that contain derivatives of unknown functions of one or more variables are known as differential equations. Differential equations also arise in the areas of Economics, Medicine, Psychology, Operations Research, Biology and Anthropology.

The development of numerical methods to obtain approximate solutions to the models becomes necessary since not all the differential equations have analytic solution; hence numerical schemes are developed to solve such differential equations.

The development of numerical methods for the initial value problem (IVP) of ordinary differential equations (ODEs) of the form

$$y''(x) = f(x, y, y'), y(a) = y_a, y'(a) = y'_a \tag{1.1}$$

$$y(a) = y_a, y'(a) = y'_a$$

Where f is continuous in $[a, b]$ and can be categorize into the general or special cases.

Equation (1.1) is usually reduced to systems of two first order ordinary differential equations. Some scholars such as Lambert [1], Kayode [2], Adee, *et al.* [3], Onumanyi, *et al.* [4], Yahaya and Badmus [5] have attempted the solution of (1.1) using Linear Multistep methods without reduction to systems of first order ordinary differential equations.

A special category of multistep methods are linear multistep methods, where the numerical solution to the ordinary differential equation at a specific location is expressed as linear combination of the numerical solution values and the function values at previous points (Nikesh [6]. Linear multistep methods have been studied by many scholars: Henrici [7], Dahlquist [8], Gear [9], Shampine and Watts [10], Fatunla [11], Amodio and Mazzia [12].

This paper is concerned with the development of Continuous Two Step Block Hybrid methods using Hermite polynomials as basis functions. We will derive the two step method with four off-grid points and the analysis of the methods will be determined.

Despite the successful application of Linear Multistep methods (LMM) of the form

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$$\sum_{j=0}^{k}\alpha_j y_{n+j} - h^2\sum_{j=0}^{k}\beta_j f_{n+j} = 0 \tag{1.2}$$

To solving problems of special second order ODEs, few methods have been proposed for solving directly the general second order IVP of the form (1.1).

We construct the continuous formulation of (1.2) in the form

$$\sum_{j=0}^{k}\alpha_j(x) y_{n+j} = h^2\sum_{j=0}^{k}\beta_j(x) f_{n+j} \tag{1.3}$$

Equation (1.3) has the ability to generate several methods which are combined and implemented in block form to solve (1.1) directly. [13-15].

A hybrid method is an extension of (1.2) which involves $f(x,y)$ evaluated at off-grid point (x_{n+v}, y_{n+v}), $0 < v < k, v \notin \{0,1,2,\ldots,k\}$ where the value of y_{n+v} is given by a separate formula. In particular, a k-step hybrid method for the second order ordinary differential equation is given by

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\beta_v f_{n+v} \tag{1.4}$$

The general form of the continuous hybrid linear multistep method with one off-grid collocation point is written in the form

$$y(x) = \sum_{j=0}^{k}\alpha_j(x) y_{n+j} + h^2\sum_{j=0}^{k}\beta_j(x) f_{n+j} + h^2\beta_v(x) f_{n+v} \tag{1.5}$$

Where $\alpha_j(x), \beta_j(x)$ and $\beta_v(x)$ are the continuous coefficients of the method, v is chosen as the midpoint of the subinterval $[x_{n+k-1}, x_{n+k}]$, k is the step number (Gear [9].

2. The Derivation of the Method

The general form of two-step linear multistep method is defined by the difference equation:

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} = h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}) \tag{2.1}$$

Let the approximation of the exact solution be given in the form

$$y(x) = \sum_{j=0}^{t+m-1} a_j H_j(x - x_n) \tag{2.2}$$

Where t is the number of interpolation points, m is the number of collocation points and $H_j(x)$ are Hermite polynomials generated by the formula:

$$H_n = (-1)^n e^{\left(\frac{x^2}{2}\right)} \frac{d^n}{dx^n} e^{\left(-\frac{x^2}{2}\right)} \tag{2.3}$$

The second derivative of (2.3) is given by

$$y''(x) = \sum_{j=0}^{t+m-1} a_j H_j''(x - x_n) \tag{2.4}$$

Where $x \in [a,b]$, the $a_{j's}$ are real unknown coefficients to be determined and t+m is the sum of interpolation and collocation points. The solution of (2.4) is determined on the partition $\pi_N: a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$ of the integration interval [a,b] with a constant step size h, given by

$$h = \frac{x_n - x_0}{N} = \frac{b-a}{N},\ n = 0,1,2,..,N$$

Substituting (2.4) into (1.1) gives

$$\sum_{j=2}^{t+m-1} a_j H_j''(x - x_n) = f(x, y, y') \tag{2.5}$$

2.1. Derivation of Two Step Block Hybrid Block Methods (TSBHM)

Consider the following specifications:

k=2, t=2 and m=6 and $v_1 = \frac{1}{2}$, $v_2 = \frac{4}{3}$, $v_3 = \frac{3}{2}$, $v_4 = \frac{5}{3}$. Then we obtained the equations

$$y(x)=\sum_{j=0}^{7}a_j H_j(x-x_n) \tag{2.6}$$

$$y''(x)=\sum_{j=0}^{7}a_j H_j''(x-x_n) \tag{2.7}$$

Equation (2.7) is the continuous methods and the modified block formula is employed to obtain the values of $y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{4}{3}}, y_{n+\frac{3}{2}}, y_{n+\frac{5}{3}}, y_{n+2}, y'_{n+\frac{1}{2}}, y'_{n+1}, y'_{n+\frac{4}{3}}, y'_{n+\frac{3}{2}}, y'_{n+\frac{5}{3}}, y'_{n+2}$ needed for implementation. This approach yield discrete schemes in the form

$$u=mx \tag{2.8}$$

Where

$$u=\left[y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{4}{3}}, y_{n+\frac{3}{2}}, y_{n+\frac{5}{3}}, y_{n+2}, y'_{n+\frac{1}{2}}, y'_{n+1}, y'_{n+\frac{4}{3}}, y'_{n+\frac{3}{2}}, y'_{n+\frac{5}{3}}, y'_{n+2}\right]^T,$$

$$x=\left[h^2 f_n, h^2 f_{n+\frac{1}{2}}, h^2 f_{n+\frac{4}{3}}, h^2 f_{n+\frac{3}{2}}, h^2 f_{n+\frac{5}{3}}, h^2 f_{n+2}, hf_n, hf_{n+\frac{1}{2}}, hf_{n+\frac{4}{3}}, hf_{n+\frac{3}{2}}, hf_{n+\frac{5}{3}}, hf_{n+2}\right]^T$$

and m is a 12×12 matrix.

3. Analysis of the Two Step Block Hybrid Method

Definition 1.1: Order and Error Constant

Following Fatunla [11], we define the local truncation error associated with the conventional form of (1.2) to be the linear difference operator:

$$L[y(x);h]=\sum_{j=o}^{k}\propto_j y(x+jh)-h^2\beta_j y''(x+jh) \tag{2.9}$$

Where the constant coefficients C_q, $q=0,1$ ... are given as follows:

$C_q=\sum_{j=o}^{k}\propto_j$

$C_1=\sum_{j=o}^{k}j\propto_j$

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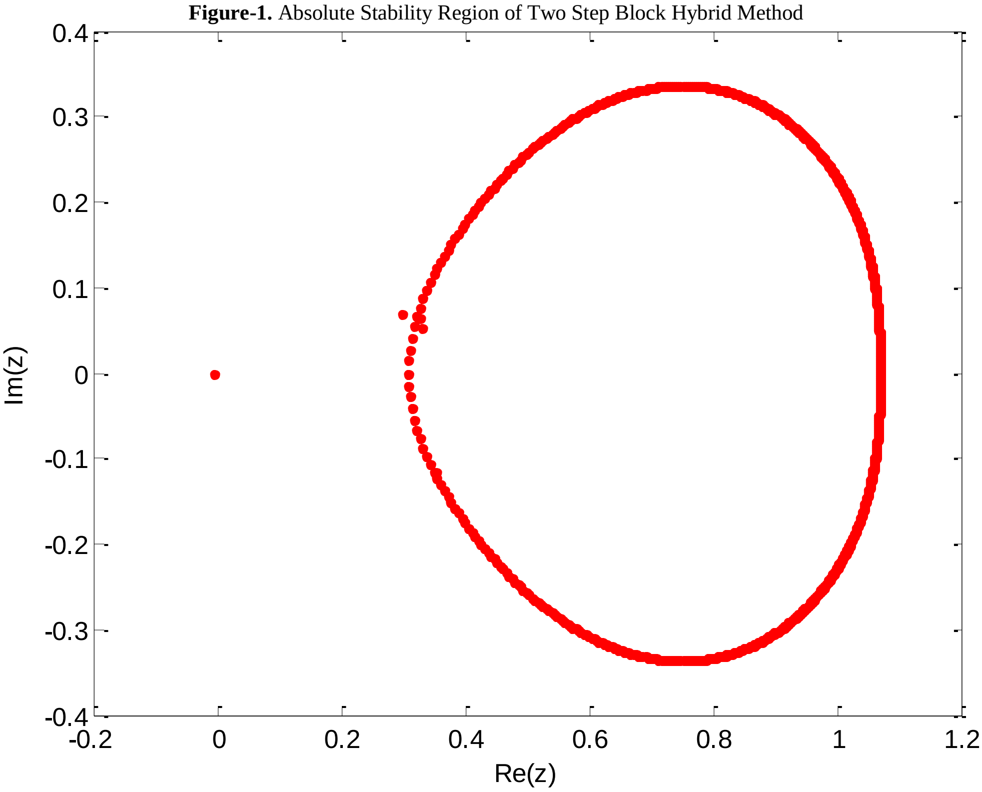
$$C_q=\frac{1}{q!}\sum_{j=o}^{k}j^q\propto_j-q(q-1)\sum_{j=o}^{k}j^{q-2}\beta_j \tag{2.10}$$

From definition (1.1), equation (2.8) is a uniform order 6 block hybrid method with error constants given as:

$$C_8=\begin{pmatrix}-\frac{151297}{42326323200},-\frac{232333}{33861058560},\frac{655607}{169305292800},\frac{561893}{72559411200},\\ \frac{423851}{28217548800},\frac{2273987}{16905292800},\frac{11029763}{16905292800},\frac{244225}{6772211712},\\ \frac{158075}{6772211712},\frac{3917639}{169305292800},\frac{3949763}{169305292800},\frac{3315971}{169305292800}\end{pmatrix}^T$$

According to Henrici [7], the block hybrid method is consistent since $p>1$, it has also been shown to be zero constant, hence convergent.

Figure-1. Absolute Stability Region of Two Step Block Hybrid Method



4. Numerical Example

Three numerical examples are solved using our block hybrid method.

1. $y'' = y'$, $y(0) = 0$, $y'(0) = -1$, $h = 0.1$

Analytical Solution given by $y(x) = 1 - \exp(x)$

2. $y'' + 1001y' + 1000y = 0$, $y(0) = 1$, $y'(0) = -1$, $h = 0.05$

Analytical Solution: $y(x) = \exp(-x)$

3. $y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0$, $y(1) = 1$, $y'(1) = 1$, $h = \frac{1}{320}$

Analytical Solution as $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$

Table-1. Computed results from the proposed methods for problem 1

X	Analytical Solution	Approx. Solution of TSBHM	Absolute Errors in TSBHM	Absolute Errors in Yahaya and Badmus (2009)
0.1	-0.10517091807565	-0.10517091807645	8.000 X 10^{-13}	8.79316 X 10^{-05}
0.2	-0.22140275816017	-0.22140275816157	1.400 X 10^{-12}	3.26718 X 10^{-04}
0.3	-0.34985880757600	-0.34985880757907	3.070 X 10^{-12}	2.21556 X 10^{-03}
0.4	-0.49182469764127	-0.49182469764582	4.550 X 10^{-12}	4.85709 X 10^{-03}
0.5	-0.64872127070013	-0.64872127070756	7.430 X 10^{-12}	9.09773 X 10^{-03}
0.6	-0.82211880039051	-0.82211880040069	1.018 X 10^{-11}	1.43914 X 10^{-02}
0.7	-1.01375270747048	-1.01375270748519	1.471 X 10^{-11}	2.14379 X 10^{-02}
0.8	-1.22554092849247	-1.22554092851166	1.919 X 10^{-11}	2.98987 X 10^{-02}
0.9	-1.45960311115695	-1.45960311118293	2.598 X 10^{-11}	4.03007 X 10^{-02}
1.0	-1.71828182845905	-1.71828182849186	3.281 X 10^{-11}	5.25521 X 10^{-02}

Table-2. Computed results from the proposed methods for problem 2

X	Analytical Solution	Approx. Solution of TSBHM	Absolute Errors in TSBHM	Absolute Errors in Jator (2007)
0.1	0.90483741803596	0.90483741803595	1.0 X 10^{-14}	6.98677 X 10^{-12}
0.2	0.81873075307798	0.81873075307795	3.0 X 10^{-14}	1.00275 X 10^{-12}
0.3	0.74081822068172	0.74081822068168	4.0 X 10^{-14}	7.85878 X 10^{-12}
0.4	0.67032004603564	0.67032004603559	5.0 X 10^{-14}	1.04778 X 10^{-11}
0.5	0.60653065971263	0.60653065971257	6.0 X 10^{-14}	6.32212 X 10^{-11}
0.6	0.54881163609403	0.54881163609396	7.0 X 10^{-14}	1.00508 X 10^{-11}
0.7	0.49658530379141	0.49658530379134	7.0 X 10^{-14}	9.36336 X 10^{-12}
0.8	0.44932896411722	0.44932896411715	7.0 X 10^{-14}	2.64766 X 10^{-12}
0.9	0.40656965974060	0.40656965974053	7.0 X 10^{-14}	1.06793 X 10^{-11}
1.0	0.36787944117144	0.36787944117137	7.0 X 10^{-14}	2.32731 X 10^{-11}

Table-3. Computed results from the proposed methods for problem 3

X	Analytical Solution	Approx. Solution of TSBHM	Absolute Errors in TSBHM	Absolute Errors in Badmus and Yahaya (2009)
1.000000	1.00000000000000	1.00000000000000	0	0
1.003125	1.00307652585770	1.00307652585769	1.0000 X 10^{-14}	3.8354 X 10^{-05}
1.006250	1.00605750308352	1.00605750308354	2.0000 X 10^{-14}	7.5004 X 10^{-05}
1.009375	1.00894499508884	1.00894499508887	3.0000 X 10^{-14}	1.0592 X 10^{-04}
1.012500	1.01174101816799	1.01174101816801	2.0000 X 10^{-14}	1.3548 X 10^{-04}
1.015625	1.01444754268641	1.01444754268643	2.0000 X 10^{-14}	1.5557 X 10^{-04}
1.018750	1.01706649423567	1.01706649423569	2.0000 X 10^{-14}	1.8637 X 10^{-04}
1.021875	1.01959975475629	1.01959975475632	3.0000 X 10^{-14}	1.9606 X 10^{-04}
1.025000	1.02204916362943	1.02204916362947	4.0000 X 10^{-14}	2.2104 X 10^{-04}
1.028125	1.02441651873840	1.02441651873844	4.0000 X 10^{-14}	2.0563 X 10^{-04}
1.031250	1.02670357750081	1.02670357750085	4.0000 X 10^{-14}	2.7791 X 10^{-04}

5. Discussion of Results

We have proposed a two step block hybrid method (TSBHM) with continuous coefficients from which multiple finite difference methods were obtained and applied as simultaneous numerical integrators, without first adapting the ODE to an equivalent first order system. The methods were derived using hermite polynomial as a basis function. We conclude that the new block hybrid method is of uniform order 6 and is suitable for direct solution of general second order differential equations. The method is self- starting and all the discrete equations used were obtained from the single continuous formulation including their derivatives which were evaluated at some interior points to form part of the block. The application of the block hybrid method on three real life numerical problems give approximate results which tend to converge to their respective analytical solutions. The absolute errors obtained from our method shows the level of convergence and accuracy of our methods.

6. Conclusion

Classes of hybrid collocation methods for the direct solution of initial value problems of general second order ordinary differential equations have been developed in this paper. The use of predictor-corrector method to solve initial value problems of general second order ordinary differential equations recorded success but with low accuracy and high computational cost. The Hermite polynomial is used to develop the schemes using collocation and interpolation techniques with the incorporation of off-grid points for the two step methods to approximate the solutions of initial value problems. The schemes are implemented as block methods, having the capacity to generate simultaneous solutions at different points in a single application of the methods. The block solution formation has made the preferred use of block methods to the use of predictor-corrector methods for second order initial value problems which is more accurate and faster than the conventional integration procedures.

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