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A Stable Approach for Numerical Differentiation by Local Regularization Method with its Regularization Parameter Selection Strategies

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Abstract

The local regularization method for solving the first-order numerical differentiation problem is considered in this paper. The *a-priori and a-posteriori* selection strategy of the regularization parameter is introduced, and the convergence rate of local regularization solution under some assumption of the exact derivative is also given. Numerical comparison experiments show that the local regularization method can reflect sharp variations and oscillations of the exact derivative while suppress the noise of the given data effectively.

Keywords: Local regularization; Numerical differentiation; Regularization parameter; Discrepancy principle.

1. Introduction

Numerical differentiation, which aims to compute the derivative of a function from its measured data approximately, has extensive application values in scientific studies and engineering practices. For example, differential operator method is the most common image edge detection method [1]. In general, image edges can be detected by finding the maximum of first-order derivatives or the zero-crossing of second-order derivatives of the image intensity [2].

Numerical differentiation is a classical ill-posed problem, and its main difficulty is the instability of numerical derivatives. There have been many stable methods developed for solving this problem. Generally speaking, these methods can be categorized as the finite difference method [3], the regularization method [4], the mollification method [5], the Lanczos integral method [6, 7] and so on.

Assume that $f \in H^1[0,1]$ and $u = f' \in L^2[0,1]$ is its first-order derivative, then f and u satisfy the following Volterra equation:

$$A[u](x) := \int_0^x u(s)ds = f(x), \quad x \in [0,1],$$

provided that $f(0) = 0$. Consider the noisy data $f^\delta \in L^2[0,1]$ of f, it is unstable to solve

$$A[u](x) = f^\delta(x), \quad x \in [0,1] \tag{1.1}$$

directly, and some regularization methods should be introduced. An immediate idea is to solve (1.1) by the Tikhonov regularization method, i.e., the regularization solution satisfies

$$ru_{Tik}(x) + A^*A[u_{Tik}](x) = A^*[f^\delta](x), \quad x \in [0,1],$$

where $r > 0$ is the regularization parameter and $A^*[u](x) := \int_x^1 u(s)ds$ is the adjoint operator of A in $L^2[0,1]$. Since the operator A is nonnegative in $L^2[0,1]$, the Volterra equation (1.1) can also be solved by Lavrentiev regularization method [8], where the regularization solution satisfies

$$ru_{Lav}(x) + A[u_{Lav}](x) = f^\delta(x), \quad x \in [0,1].$$

As we know, the solution of Tikhonov regularization is too smooth, while the solution of Lavrentiev regularization is too sensitive to the noise. In order to ensure the calculation accuracy while suppressing the noise, some eclectic methods should be introduced.

The local regularization method can be used to solve Volterra integral equation, Fredholm integral equation of the first kind [9-12]. If only consider the equation (1.1), the local regularization method adopts the information of

$f^{\delta}(x)$ on a small future interval $[x, x+r]$ when we compute $u(x)$, which avoids the overuse of $f^{\delta}(x)$ in Tikhonov regularization and the underuse of $f^{\delta}(x)$ in Lavrentiev regularization. Therefore, the local regularization method will be adopted to solve the Volterra equation (1.1). In this paper, we give the *a-priori* and *a-posteriori* parameter selection strategy in $L^2[0,1]$ space with the convergence rate of local regularization solution, where the *a-priori* assumption is u belongs to Sobolev space $H^{\alpha}[0,1], \alpha \in (0,1)$. The *a-posteriori* parameter selection strategy we used is the extension of the generalized discrepancy principle for Lavrentiev regularization method given in [13-15].

The paper is organized as follows. In Section 2, the local regularization method for solving numerical differentiation problem is given. The *a-priori* and *a-posteriori* selection strategy of the regularization parameter with the convergence rate of local regularization solution are given in Section 3. At last, numerical comparison experiments are shown in Section 4.

2. Local Regularization for Numerical Differentiation

Let $r \in (0, R]$, where $R > 0$ be a small fixed constant and extend the domain of f and u to $[0, 1+R]$, then it has

$$\int_0^{x+\rho} u(s)ds = f(x+\rho), \quad x \in [0,1], \quad \rho \in [0,r]. \tag{2.1}$$

Splitting the integral in (2.1) yields

$$\int_0^{x} u(s)ds + \int_x^{x+\rho} u(s)ds = f(x+\rho), \quad x \in [0,1], \quad \rho \in [0,r].$$

Integrating both sides of the equation with respect to ρ on $[0, r]$ yields

$$\int_0^r \int_0^x u(s)dsd\rho + \int_0^r \int_x^{x+\rho} u(s)dsd\rho = \int_0^r f(x+\rho)d\rho, \quad x \in [0,1],$$

i.e.,

$$r\int_0^x u(s)ds + \int_0^r \int_0^{\rho} u(x+s)dsd\rho = \int_0^r f(x+\rho)d\rho, \quad x \in [0,1]. \tag{2.2}$$

Notice that equation (2.2) still is an equation that the exact solution $u = f'$ satisfies exactly. In the text that follows, we assume that $f \in H^1[0, 1+R]$, $u \in L^2[0, 1+R]$ and $f^{\delta} \in L^2[0, 1+R]$ satisfying

$$\| f^{\delta} - f \|_{L^2[0,1+R]} \le \delta. \tag{2.3}$$

In order to solve (1.1) stably and obtain a desirable approximate derivative, we replace $u(x+s)$ in the second term of (2.2) by $u(x)$ as $0 < r < R$ is small enough, and thus

$$r\int_0^x u^{r,\delta}(s)ds + \frac{r^2}{2} u^{r,\delta}(x) = \int_0^r f^{\delta}(x+\rho)d\rho, \quad x \in [0,1],$$

i.e.,

$$A[u^{r,\delta}](x) + \frac{r}{2} u^{r,\delta}(x) = f_r^{\delta}(x), \quad x \in [0,1], \tag{2.4}$$

Where $f_r^{\delta}(x) := \frac{1}{r}\int_0^r f^{\delta}(x+\rho)d\rho$. Since the operator A is nonnegative in $L^2[0,1]$, the equation (2.4) has a unique solution $u^{r,\delta} \in L^2[0,1]$, which depends continuously on f_r^{δ}.

3. The Selection Strategy of Regularization Parameter

In the following, we first introduce the *a-priori* selection strategy of the regularization parameter r and the convergence rate of the regularization solution $u^{r,\delta}$. For the simplicity of notation, the norm $\|\cdot\|$ without the subscript means the norm of $L^2[0,1]$ deduced by L^2 inner product $(\cdot,\cdot)$.

Theorem 1 Assume that $u \in H^{\alpha}[0, 1+R], \alpha \in (0,1)$ with $\| u \|_{H^{\alpha}[0,1+R]} \le M$, $f^{\delta} \in L^2[0, 1+R]$ satisfying (2.3) is the noise data and $u^{r,\delta}$ is the unique solution of (2.4), then it has the error estimate

$$\| u^{r,\delta} - u \| \le \frac{2\delta}{r} + \frac{M}{\alpha+1} r^{\alpha}, \tag{3.1}$$

and the optimal convergence rate $\| u^{r,\delta} - u\| = O(\delta^{\frac{\alpha}{\alpha+1}})$ holds when $r = [\frac{2\delta}{M}(1+\frac{1}{\alpha})]^{\frac{1}{\alpha+1}}$.

Proof: Denote $u^r(x)$ as the solution of

$$A[u^r](x) + \frac{r}{2}u^r(x) = f_r(x), \quad x \in [0,1], \tag{3.2}$$

where $f_r(x) = \frac{1}{r}\int_0^r f(x+\rho)d\rho$, then it has

$$\| u^{r,\delta} - u\| \le \| u^{r,\delta} - u^r\| + \| u^r - u\|. \tag{3.3}$$

Since

$$((\frac{r}{2}I + A)u, u) = \frac{r}{2}(u,u) + (Au,u) \ge \frac{r}{2}(u,u),$$

it has $\|(\frac{r}{2}I + A)u\| \ge \frac{r}{2}\| u\|$, and thus $\|(\frac{r}{2}I + A)^{-1}\| \le \frac{2}{r}$. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
\| f_r^\delta - f_r\|^2 &= \frac{1}{r^2}\int_0^1 \{\int_0^r [f^\delta(x+\rho) - f(x+\rho)]d\rho\}^2 dx \\
&\le \frac{1}{r^2}\int_0^1 r\int_0^r [f^\delta(x+\rho) - f(x+\rho)]^2 d\rho dx \\
&= \frac{1}{r}\int_0^r \int_\rho^{1+\rho} [f^\delta(x) - f(x)]^2 dx d\rho \le \delta^2.
\end{aligned}$$

Hence, it has

$$\| u^{r,\delta} - u^r\| = \|(\frac{r}{2}I + A)^{-1}(f_r^\delta - f_r)\| \le \frac{2}{r}\| f_r^\delta - f_r\| \le \frac{2\delta}{r}. \tag{3.4}$$

Rewriting the equation (2.2) as

$$A[u](x) + \frac{r}{2}u(x) = f_r(x) + \frac{r}{2}u(x) - \frac{1}{r}\int_0^r\int_0^\rho u(x+s)ds d\rho, \quad x \in [0,1], \tag{3.5}$$

and subtracting (3.2) by (3.5) yield

$$\begin{aligned}
(\frac{r}{2}I + A)[u^r(x) - u(x)] &= \frac{1}{r}\int_0^r\int_0^\rho u(x+s)ds d\rho - \frac{r}{2}u(x) \\
&= \frac{1}{r}\int_0^r\int_0^\rho [u(x+s) - u(x)]ds d\rho.
\end{aligned}$$

Hence, it has

$$u^r(x) - u(x) = \frac{1}{r}(\frac{r}{2}I + A)^{-1}\int_0^r\int_0^\rho [u(x+s) - u(x)]ds d\rho, \tag{3.6}$$

and then

$$\begin{aligned}
\| u^r - u\|^2 &\le \frac{4}{r^4}\| \int_0^r\int_0^\rho [u(x+s) - u(x)]ds d\rho\|^2 \\
&\le \frac{4}{r^4}\int_0^1 \{\int_0^r\int_0^\rho [u(x+s) - u(x)]ds d\rho\}^2 dx \\
&\le \frac{4}{r^4}\int_0^1 r\int_0^r \{\int_0^\rho [u(x+s) - u(x)]ds\}^2 d\rho dx \\
&\le \frac{4}{r^3}\int_0^1\int_0^r\int_0^\rho s^{2\alpha+1}ds \cdot \int_0^\rho \frac{[u(x+s) - u(x)]^2}{s^{2\alpha+1}} ds d\rho dx \\
&\le \frac{4}{r^3}\int_0^r \frac{\rho^{2\alpha+2}}{2\alpha+2} \cdot \int_0^1\int_0^\rho \frac{[u(x+s) - u(x)]^2}{s^{2\alpha+1}} ds dx d\rho,
\end{aligned}$$

where the Cauchy-Schwarz inequality is used twice. Since $u \in H^\alpha[0,1+R]$ and $\| u\|_{H^\alpha[0,1+R]} \le M$, i.e.,

$$\int_0^{1+R}\int_0^{1+R} \frac{[u(x) - u(y)]^2}{|x-y|^{2\alpha+1}} dx dy \le M^2,$$

it has

$$\int_0^1\int_0^\rho \frac{[u(x+s)-u(x)]^2}{s^{2\alpha+1}}dsdx \le M^2.$$

Hence, it has

$$\| u^r - u\|^2 \le \frac{4M^2}{(2\alpha+2)(2\alpha+3)} r^{2\alpha},$$

i.e.,

$$\| u^r - u\| \le \frac{2M}{\sqrt{(2\alpha+2)(2\alpha+3)}} r^\alpha < \frac{M}{\alpha+1} r^\alpha. \tag{3.7}$$

It follows from (3.3), (3.4) and (3.7) that

$$\| u^{r,\delta} - u\| \le \frac{2\delta}{r} + \frac{M}{\alpha+1} r^\alpha.$$

Hence, the optimal convergence rate of $u^{r,\delta}$ is $\| u^{r,\delta} - u\| = O(\delta^{\frac{\alpha}{\alpha+1}})$ when $r = [\frac{2\delta}{M}(1+\frac{1}{\alpha})]^{\frac{1}{\alpha+1}}$.
□

The *a-priori* selection strategy of the regularization parameter given in Theorem 1 relies on the *a-priori* assumption of exact solution $u = f'$, which is usually unknown in many practical problems. Compared with the *a-priori* selection strategy, the *a-posteriori* selection strategy generally only relies on the given data and its noise level, therefore, is more useful. There have been many *a-posteriori* choice strategies of the regularization parameter, such as the discrepancy principle and its generalizations [13, 16], the generalized cross-validation method [17], the L-curve criterion [18] and the Arcangeli's method [19].

Next, we will extend the generalized discrepancy principle for Lavrentiev regularization method [13-15] to local regularization method. The principle determines the regularization parameter $r = r(\delta)$ by solving the nonlinear equation

$$\| A[u^{r,\delta}] - f_r^\delta\| = C\delta^\nu, \tag{3.8}$$

where $C > 0$ and $0 < \nu < 1$ are two given constants. In consideration of the solvability of (3.8), we have the following conclusion.

Theorem 2 Assume that $\| f_R^\delta\| \ge \delta$ and $\delta < C\delta^\nu \le \frac{R}{R+2}\| f_R^\delta\|$ for two constants $C > 0$ and $0 < \nu < 1$, then the equation (3.8) has at least one solution $r \in (0, R]$.

Proof: Since $u^{r,\delta}$ is the solution of (2.4), it has

$$\| f_r^\delta\| = \|(\frac{r}{2}I + A)u^{r,\delta}\| \le (\frac{r}{2}+1)\| u^{r,\delta}\|,$$

where $\| A\| \le 1$ is used, and thus

$$\| A[u^{r,\delta}] - f_r^\delta\| = \frac{r}{2}\| u^{r,\delta}\| \ge \frac{r}{r+2}\| f_r^\delta\|, \quad r \in (0, R].$$

When $r = R$, assumption condition shows that

$$\| A[u^{R,\delta}] - f_R^\delta\| \ge \frac{R}{R+2}\| f_R^\delta\| \ge C\delta^\nu. \tag{3.9}$$

Moreover, the triangle inequality yields

$$\| u^{r,\delta}\| \le \| u^{r,\delta} - u^r\| + \| u^r - u\| + \| u\|. \tag{3.10}$$

According to (3.6), it has

$$\begin{aligned}
\| u^r - u\|^2 &\le \frac{4}{r^4}\|\int_0^r\int_0^\rho [u(x+s)-u(x)]dsd\rho\|^2 \\
&\le \frac{4}{r^4}\int_0^1\{\int_0^r\int_0^\rho [u(x+s)-u(x)]dsd\rho\}^2 dx \\
&\le \frac{4}{r^4}\int_0^1 r\int_0^r\{\int_0^\rho \left[u(x+s)-u(x)\right]ds\}^2 d\rho dx \\
&\le \frac{4}{r^4}\int_0^1 r\int_0^r \rho\int_0^\rho [u(x+s)-u(x)]^2 dsd\rho dx
\end{aligned}$$

$$\le \frac{8}{r^4}\int_0^1 r\int_0^r \rho[\int_0^\rho u^2(x+s)ds + \int_0^\rho u^2(x)ds]d\rho dx.$$

Denote

$$I_1 = \frac{8}{r^4}\int_0^1 r\int_0^r \rho\int_0^\rho u^2(x+s)dsd\rho dx,\quad I_2 = \frac{8}{r^4}\int_0^1 r\int_0^r \rho\int_0^\rho u^2(x)dsd\rho dx,$$

then it has $I_2 = \frac{8}{3}\int_0^1 u^2(x)dx \le \frac{8}{3}\| u\|^2_{L^2[0,1+R]}$ and

$$I_1 \le \frac{8}{r^4}\int_0^1 r\int_0^r \rho\int_0^r u^2(x+s)dsd\rho dx$$

$$\le \frac{4}{r}\int_0^1\int_0^r u^2(x+s)dsdx = \frac{4}{r}\int_0^1\int_x^{x+r} u^2(t)dtdx$$

$$= \frac{4}{r}[\int_0^r\int_0^t u^2(t)dxdt + \int_r^1\int_{t-r}^t u^2(t)dxdt + \int_1^{1+r}\int_{t-r}^1 u^2(t)dxdt$$

$$\le \frac{4}{r}\int_0^{1+r} ru^2(t)dt \le 4\| u\|^2_{L^2[0,1+R]}.$$

Estimations of I_1 and I_2 show that

$$\| u^r - u\| \le 3\| u\|_{L^2[0,1+R]}. \tag{3.11}$$

According to (3.4), (3.10) and (3.11), it has

$$\| A[u^{r,\delta}] - f_r^\delta\| = \frac{r}{2}\| u^{r,\delta}\| \le \frac{r}{2}(\frac{2\delta}{r} + 3\| u\|_{L^2[0,1+R]} + \| u\|) \le \delta + 2r\| u\|_{L^2[0,1+R]},$$

and thus

$$\lim_{r\to 0}\| A[u^{r,\delta}] - f_r^\delta\| \le \delta. \tag{3.12}$$

Inequalities (3.9), (3.12) and the continuity of $\| A[u^{r,\delta}] - f_r^\delta\|$ show that the equation (3.8) has at least one solution $r\in(0,R]$. □

Since the equation (3.8) may have more than one solution on the interval $(0,R]$, we determine the *a-posteriori* selection strategy of $r = r(\delta)$ as

$$r(\delta) := \inf\{r\in(0,R] \,|\, \| A[u^{r,\delta}] - f_r^\delta\| = C\delta^\nu\}, \tag{3.13}$$

where C and ν satisfy assumption conditions given in Theorem 2. Next, we will give the *a-posteriori* convergence rate of the regularization solution $u^{r(\delta),\delta}$ under some assumptions.

Theorem 3 Assume that conditions of Theorem 1 and Theorem 2 hold, and there exists a positive number ϵ_0 satisfying $\| f_r\| \ge \epsilon_0, r\in(0,R]$, then for the regularization solution $u^{r(\delta),\delta}$ with $r = r(\delta)$ satisfying (3.13), it has

$$\| u^{r(\delta),\delta} - u\| = O(\delta^{min\{1-\nu,\nu\alpha\}}),$$

and the optimal convergence rate $\| u^{r(\delta),\delta} - u\| = O(\delta^{\frac{\alpha}{\alpha+1}})$ holds when $\nu = \frac{1}{\alpha+1}$.

Proof: The triangle inequality yields

$$\| u^{r(\delta),\delta} - u\| \le \| u^{r(\delta),\delta} - u^{r(\delta)}\| + \| u^{r(\delta)} - u\| \le \frac{2\delta}{r(\delta)} + \| u^{r(\delta)} - u\|, \tag{3.14}$$

where $u^{r(\delta)}$ is the regularization solution satisfying (3.2) with $r = r(\delta)$. Notice that

$$C\delta^\nu = \| A[u^{r(\delta),\delta}] - f^\delta_{r(\delta)}\| \ge \frac{r(\delta)}{r(\delta)+2}\| f^\delta_{r(\delta)}\|$$

$$\ge \frac{r(\delta)}{r(\delta)+2}(\| f_{r(\delta)}\| - \| f^\delta_{r(\delta)} - f_{r(\delta)}\|)$$

$$\ge \frac{r(\delta)}{r(\delta)+2}(\epsilon_0 - \delta) > \frac{r(\delta)}{r(\delta)+2}(\epsilon_0 - C\delta^\nu),$$

hence it has

$$r(\delta) \le \frac{2C\delta^{\nu}}{\epsilon_0 - 2C\delta^{\nu}}, \tag{3.15}$$

when δ is so small that $2C\delta^{\nu} < \epsilon_0$ is satisfied, and thus $\lim_{\delta\to 0} r(\delta) = 0$.
Since

$$\begin{aligned}\| u^{r(\delta),\delta} \| &\le \| u^{r(\delta),\delta} - u^{r(\delta)} \| + \| u^{r(\delta)} \| \\ &\le \frac{2\delta}{r(\delta)} + \| u^{r(\delta)} - u \| + \| u \| \\ &\le \frac{\delta^{1-\nu}}{C} \| u^{r(\delta),\delta} \| + 3\| u \|_{L^2[0,1+R]} + \| u \|,\end{aligned}$$

it has

$$\| u^{r(\delta),\delta} \| \le \frac{4C\| u \|_{L^2[0,1+R]}}{C - \delta^{1-\nu}},$$

and thus

$$\frac{2\delta}{r(\delta)} = \frac{\delta^{1-\nu}}{C} \| u^{r(\delta),\delta} \| \le \frac{4\| u \|_{L^2[0,1+R]}}{C - \delta^{1-\nu}} \delta^{1-\nu}. \tag{3.16}$$

It follows from (3.7), (3.14), (3.15) and (3.16) that

$$\| u^{r(\delta),\delta} - u \| \le \frac{4\| u \|_{L^2[0,1+R]}}{C - \delta^{1-\nu}} \delta^{1-\nu} + \frac{M}{\alpha+1} \left(\frac{2C}{\epsilon_0 - 2C\delta^{\nu}}\right)^{\alpha} \delta^{\nu\alpha},$$

i.e.,

$$\| u^{r(\delta),\delta} - u \| = O(\delta^{min\{1-\nu,\nu\alpha\}}). \tag{3.17}$$

Hence, the optimal convergence rate of $u^{r(\delta),\delta}$ is $\| u^{r(\delta),\delta} - u \| = O(\delta^{\frac{\alpha}{\alpha+1}})$ when $\nu = \frac{1}{\alpha+1}$.
□

From Theorem 3 we know that when the regularization parameter is chosen by the *a-posteriori* selection strategy (3.13), the regularization solution has the same convergence rate as the *a-priori* selection strategy given in Theorem 1.

4. Numerical Experiments

In this section, we give two numerical examples in order to show the validity of local regularization for numerical differentiation, where the local regularization solution, i.e., the solution of (2.4) is denoted as u_{Loc}.

Example 1 Consider a function

$$f(x) = \begin{cases} -\frac{1}{2}x^2 + \frac{x}{2}, & x \in [0, \frac{1}{2}], \\ \frac{1}{2}x^2 - \frac{x}{2} + \frac{1}{4}, & x \in (\frac{1}{2}, 1], \end{cases}$$

then it has

$$u(x) = f'(x) = \begin{cases} \frac{1}{2} - x, & x \in [0, \frac{1}{2}], \\ x - \frac{1}{2}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

The noise data are generated by

$$f^{\delta}(x) = f(x) + \delta * rand(x), \quad x \in [0,1],$$

where $rand(x)$ is a random function that follows the standard uniform distribution on the open interval $(0,1)$.

Firstly, let us compare the numerical results of three different regularization methods: Tikhonov, Lavrenitev and local regularization method. In Figure 1, the relative errors of u_{Tik}, u_{Lav} and u_{Loc} are shown for different error levels of the noise data, where $Rel_{\delta} = \| f^{\delta} - f \| / \| f \|$ means the relative error of f^{δ}, and Rel_u means the relative error of regularization solution. In the experiment, all the regularization parameters $r = r(\delta)$ are chosen as the ones that minimize the error of regularization solutions, i.e.,

$$r(\delta) = \arg\min_{r>0} \| u^{r,\delta} - u \|,$$

where $u^{r,\delta}$ means the regularization solution u_{Tik}, u_{Lav} or u_{Loc}. From Figure 1 we can see that the local regularization method is better than the other two methods for numerical differentiation. When the relative error of noise data is $Rel_{\delta} = 1\%$, the regularization solutions u_{Tik}, u_{Lav}, u_{Loc} and the exact solution u are shown in Figure 2, from which we can see that the local regularization method can reflect sharp variations of exact derivatives while suppress the noise effectively. As we know, the results of u_{Tik} near $x = 1$ and u_{Lav} near $x = 0$ are terrible for solving Volterra integral equation (1.1), just as Figure 2 shows. When $Rel_{\delta} = 1\%$, we solve the *a-posteriori* selection strategy (3.13) approximately, and get the regularization parameter $r = 0.052$, where the constants are chosen as $\nu = 0.5$, $C = \frac{R\| f_R^{\delta} \|}{R+2} \delta^{-\nu}$ and $R = 0.1$. As we shown in Figure 3, the result of the *a-posteriori* selection strategy is acceptable, although it is not optimal.

Example 2 Consider a function

$$f(x) = sin((\pi x)^k),$$

where k is nonnegative integer, then it has $u(x) = f'(x) = k\pi^k x^{k-1} cos((\pi x)^k)$. The noise data $f^{\delta}(x)$ is generated by the same way in Example 1.

In this example, the integer k reflects the oscillation of $u(x)$, and the bigger values of k, the stronger oscillations of $u(x)$. In table 1, the relative errors of u_{Tik}, u_{Lav} and u_{Loc} are shown for different k when the relative error of noise data is $Rel_{\delta} = 5\%$. From Table 1 we can see that the superiority of local regularization method over the other two methods, and still the stability of u_{Loc} with respect to different oscillations of $f(x)$. When $Rel_{\delta} = 5\%$, the regularization solution u_{Loc} and the exact solution u are shown in Figure 4 when $k = 2$ and $k = 4$, from which we can see that the local regularization method can reflect oscillations of exact derivatives well.

Figure-1. Relative errors of u_{Tik}, u_{Lav} and u_{Loc} for different relative error levels Rel_{δ} of the noise data

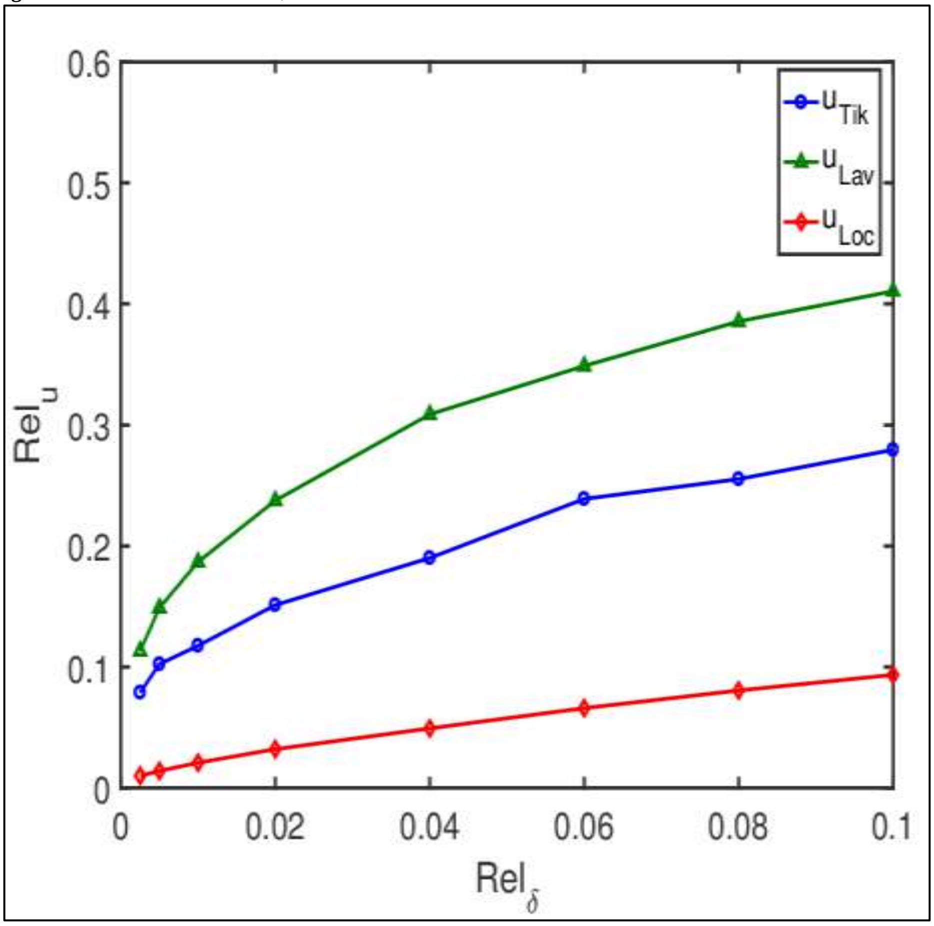

Figure-2. The comparison between the exact solution u and the regularization solution u_{Tik} (left), u_{Lav} (middle), u_{Loc} (right) when $Rel_\delta = 1\%$

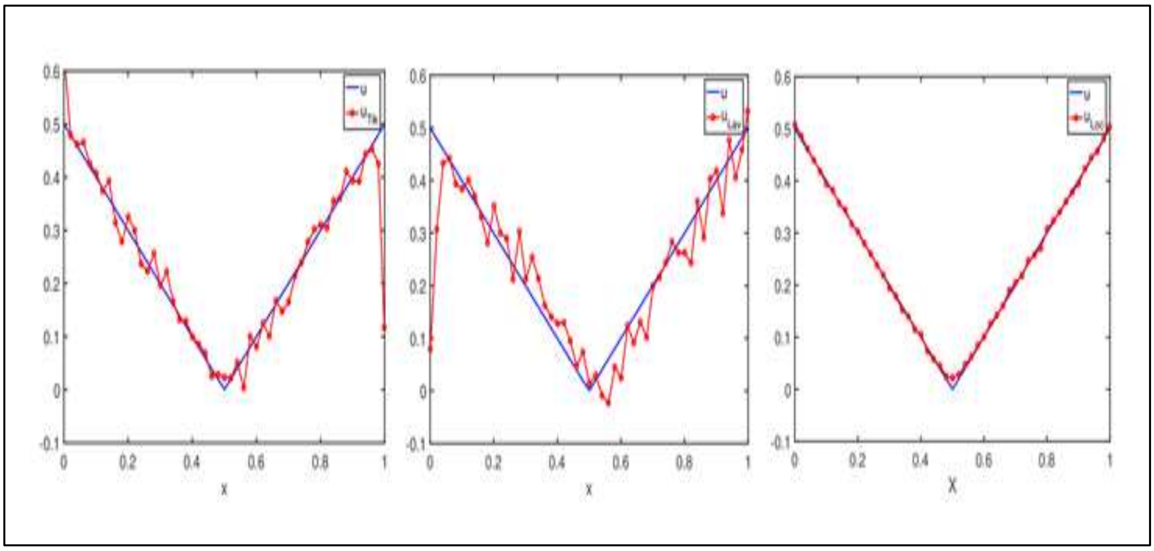

Figure-3. The relative error of u_{Loc} for different r when $Rel_\delta = 1\%$, where $r = 0.052$ is the solution of the *a-posteriori* selection strategy (3.13)

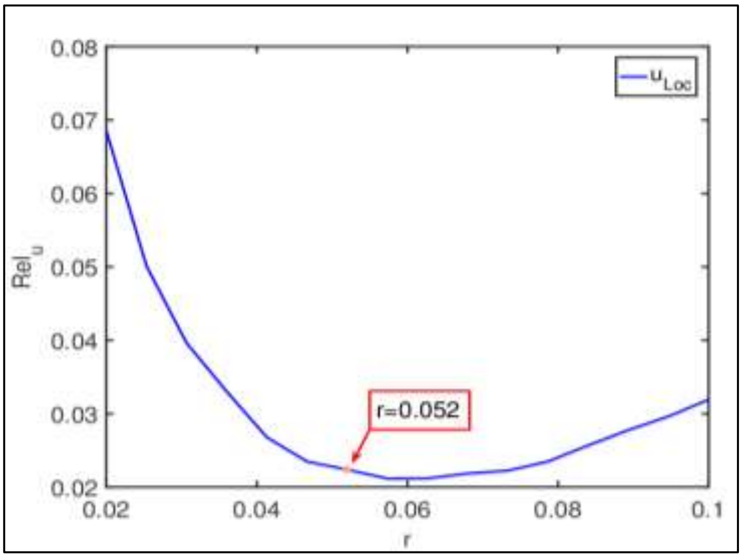

Figure-4. The comparison between the exact solution u and the regularization solution u_{Loc} when $k = 2$ (left), $k = 4$ (right) and $Rel_\delta = 5\%$

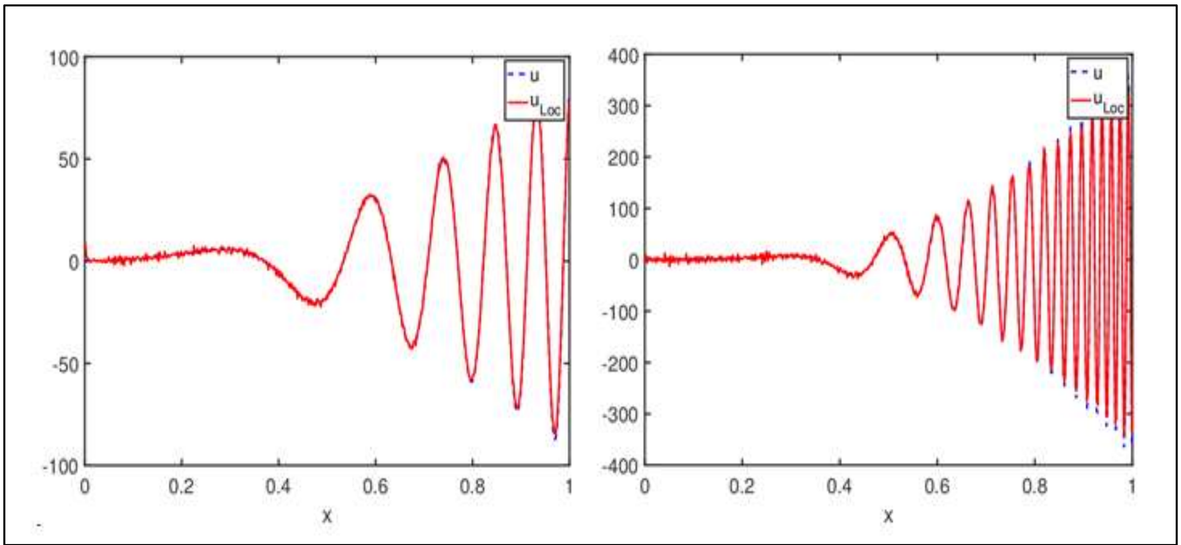

Table-1. Relative errors of u_{Tik}, u_{Lav} and u_{Loc} for different k when $Rel_{\delta} = 5\%$.

k	u_{Tik}	u_{Lav}	u_{Loc}
$k=1$	0.1477	0.2722	0.0435
$k=2$	0.1443	0.2328	0.0492
$k=3$	0.1204	0.2229	0.0463
$k=4$	0.1068	0.1172	0.0575

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