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Simple Finite-Dimensional Modules and Monomial Bases from the Gelfand-Testlin Patterns

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Abstract

One of the most important classes of Lie algebras is sl_n, which are the $n \times n$ matrices with trace 0. The representation theory for sl_n has been an interesting research area for the past hundred years and in it the simple finite-dimensional modules have become very important. They were classified and Gelfand and Tsetlin actually gave an explicit construction of a basis for every simple finite-dimensional module. This paper extends their work by providing theorems and proofs, and constructs monomial bases of the simple module.

Keywords: Finite-dimensional; Module; Irreducible; Representation; Monomial basis.

1. Introduction

Let K_n be a Lie algebra of all matrices of order n. In this paper, we work with finite-dimensional modules and hence finite-dimensional representation of sl_n. This means for $g \in sl_n$, there exists a matrix G of order N defined in such a way that

$$\text{if } g \to G \text{ and } f \to F, \text{then } \lambda g + \mu f \to \lambda G + \mu F \text{ and } [g, f] \to [G, F].$$

Choose integers $m_1, m_2, \cdots, m_n$ such that the inequality $m_1 \geq m_2 \geq \cdots \geq m_n$ is satisfied. These partitions are quite important because they appear to be the core in constructing representations. These chosen integers are used to construct some index set ξ (the explicit construction of this index set will be given in the next section). For a Lie algebra with order n, we could construct at least $\frac{n(n-1)}{2}$ possible number of such ξ with entries from a given partition. An example will be given in the next section.

Let $e_{i,j}$ be a matrix of order n which has 1 at the intersection of the i^{th} row and the j^{th} column and zeros in all other places and let E_{ij} be the matrix of order N from K_n. Note that E_{ij}, under our consideration corresponds to elements $e_{i,j} \in K_n$. It is easy to see that each matrix E_{ij}forms a linear combination of $e_{i,j}$; that is $E_{i,j} = \sum_{i,j} a_{i,j}\, e_{i,j}$ for some $a_{i,j}$. Therefore, the set $E_{i,j}$distinctly defines some representation. One could find all such representations by explicitly describing all linear transformations $E_{i,j}$.

The quest for irreducible representations of special linear algebra sl_n was reformulated: one needs matrices $E_{i,j}$ of order N satisfying the following bracket relations:

$$[E_{i,j}, E_{j,l}] = E_{i,l} \text{ when } i \neq l,$$
$$[E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j},$$
$$[E_{i_1,j_1}, E_{i_2,j_2}] = 0 \text{ when } j_1 \neq i_2 \text{ and } i_1 \neq j_2.$$

For irreducibility, the system $E_{i,j}$ is required to have no invariant subspaces.

The representation theory of sl_n has a unique nature in choosing a partition. For the classification of simple finite dimensional modules, one sets the last choice $m_n = 0$ in the partition. This controls differences between subsequent choices in a partition.

A comprehensive theory of infinitesimal transformations was first given by a Norwegian mathematician, Sophus Lie (1842-1899). I. M. Gelfand and M. L. Tsetlin gave an explicit construction of a basis for every simple finite-dimensional module of sl_n. In their work, they gave all the irreducible representations of general linear algebra (gl_n) but without theorems [1]. Recently, V. Futorny, D. Grantcharov and L. E. Ramirez provided a classification and explicit bases of tableaux of all irreducible generic Gelfand-Tsetlin modules for the Lie algebra gl_n [2]. In 2016, V. Futorny, D. Grantcharov, and L. E. Ramirez initiated the systematic study of a large class of non-generic Gelfand-Tsetlin modules - the class of $1-$singular Gelfand-Tsetlin modules. An explicit tableaux realization and the action of gl_n on these modules was provided using a new construction which they call derivative tableaux. Their construction of $1-$singular modules provides a large family of new irreducible Gelfand-Tsetlin modules of gl_n, and is a part of the classification of all such irreducible modules for $n = 3$ [3].

This paper will show that the Gelfand-Tsetlin constructions given in the year 1950 [1] forms all the irreducible representations of special linear algebra sl_n by providing proofs to results. It will also show that sl_n $-$module is simple and also construct monomial basis from these modules. Section 2 discusses some previous work and gives some notations and Section 3 presents proofs to results and shows that sl_n $-$module is simple. Then a conclusion is drawn in Section 4.

2. Notations and Preliminaries

Definition 1 (Upper Triangular Matrix). This is a matrix with entries (i,j) where $i \geq j$ are zeros.

From now on, we will denote an upper triangular matrix by $E_{i,j}$ such that entry (i,j) has a 1 and all others are zeros. Let u^+ be the set of all upper triangular matrices. If $E_{i_1,j_1}, E_{i_2,j_2} \in u^+$, then $[E_{i_1,j_1}, E_{i_2,j_2}] \in u^+$. Therefore, $u^+$is a Lie algebra and $E_{i,j}$, $i < j$ is a basis of u^+. So $E_{i,i+1}$ acts by zero. Hence $\{E_{i,i+1}\}$ are generators of u^+. We will denote a sequence of upper triangular matrices by $E^{\underline{a}}$ and a sequence of upper triangular matrices in relation to ξ by $E^{\underline{a}(\xi)}$.

Definition 2 (Lower Triangular Matrix). This is a matrix with entries (i,j) where $i \leq j$ are zeros.

Similarly, from now on, we will denote a lower triangular matrix by $F_{i,j}$ such that entry (i,j) has a 1 and all others are zeros. Let u^- be the set of all lower triangular matrices. If $F_{i_1,j_1}, F_{i_2,j_2} \in u^-$, then $[F_{i_1,j_1}, F_{i_2,j_2}] \in u^-$. Therefore, u^- is a Lie algebra and $F_{i,j}, i > j$ is a basis of u^-, with $F_{i+1,i}$ acting by zero, so $\{F_{i+1,i}\}$ are generators of u^-. Similarly, we will denote a sequence of lower triangular matrices by $F^{\underline{a}}$ and a sequence of lower triangular matrices in relation to ξ by $F^{\underline{a}(\xi)}$.

Definition 3 (Diagonal Matrix). This is a matrix with some non-zero entries on its diagonal while all other entries away from the diagonal are zero.

It is well known that the entries of the diagonals of such a square matrix are the eigenvalues. Let h be the set of all diagonal matrices with trace zero. For $H_{i_1,j_1}, H_{i_2,j_2} \in h$, $[H_{i_1,j_1}, H_{i_2,j_2}] = 0 \in h$. So $\{H_{i,i} - H_{i+1,i+1} | 1 \leq i \leq n - 1\}$ is a basis of h. Suppose $h^\star = \{\varphi: h \to \mathbb{C} | \varphi \text{is linear}\}$. The map φ is defined by giving the image of $H_{i,i} - H_{i+1,i+1}$, for all i. By definition,

$$\begin{aligned} \omega: h &\to \mathbb{C}, \\ H_{i,i} - H_{i+1,i+1} &\to 1, \\ H_{j,j} - H_{j+1,j+1} &\to 0, \forall j \neq i. \end{aligned}$$

Since ω_i generates all of h, then $\{\omega_i\}$ is a basis of $h^\star$.

Definition 4 (Representation [4]). Suppose L is a Lie algebra and let $x, y \in L$. The operation

$$\rho: L \to \text{End}(K_n)$$
$$\rho([x,y]) = [\rho(x), \rho(y)] \equiv \rho(x)\rho(y) - \rho(y)\rho(x).$$

is a Lie algebra representation. The vector space K_n is the representation space. The bracket $[\cdot,\cdot]$ is bilinear and also an endomorphism. That means

$$[\cdot,\cdot]: \text{End}(K_n) \times \text{End}(K_n) \to \text{End}(K_n).$$

It is easy to see that the Lie algebra $sl_n = u^- \oplus h \oplus u^+$. If K_n is a finite dimensional sl_n $-$module, then $H \in h$ (where $H = \oplus_i^n H_i$) acts on $K_n$such that

$$K_n = H_1 \cdot \xi_1 + \cdots + H_n \cdot \xi_n = \bigoplus_\lambda (K_n)_\lambda,$$

where λ runs over H^* (a dual) and

$(K_n)_\lambda = \{r \in (K_n) | H \cdot \xi = \lambda(H)\xi \;\; \forall H \in h\}$.

The weight spaces $(K_n)_\lambda$ are infinitely many and different from zero when $K_n$is infinite dimensional. $(K_n)_\lambda$ is called a weight space, ξ a weight vector and we called λ a weight of K_n. A highest weight vector (maximal vectors) in sl_n $-$module is a non-zero weight vector β in weight space $(K_n)_\lambda$ annihilated by the action of all upper triangular matrices. We will prove in this paper that a highest weight vector is indeed maximal and hence a generator.

The index set, ξ is an interesting construction and we will show how it is built. K_n is a vector space with bases ξ [1]. These bases depend on the choice of integer partition

$$m_1, m_2, \cdots, m_n \text{ with } (m_1 \geq m_2 \geq \cdots \geq m_n).$$

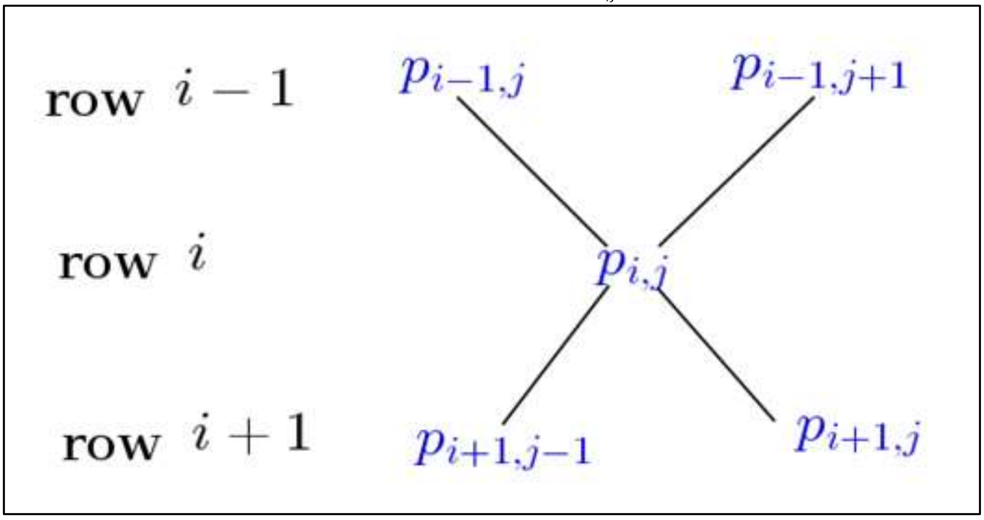
$$\xi = \begin{pmatrix} p_{1,i}, i = 1, \ldots, n-1 \\ p_{2,i}, i = 1, \ldots, n-2 \\ \ddots \quad\quad \ddots \\ p_{j,i}, \begin{cases} 1 \leq j \leq n-1 \\ 0 \leq i \leq n-j \end{cases} \end{pmatrix}. \quad\quad (1)$$

In order to understand the construction of this basis vector quite well, let us consider rows $(i-1), i$ and $(i+1)$ and entry $p_{i,j}$ in ξ. For all $p_{i,j}$, if $j < 1$ or $j > n - i$, then $p_{i,j}$ is not an entry in the index set. Otherwise, the relations of the three rows and specifically the entry $p_{i,j}$ are

$$\begin{cases} p_{i-1,j} \geq p_{i,j} \geq p_{i-1,j+1}, \\ p_{i+1,j-1} \geq p_{i,j} \geq p_{i+1,j}, \\ \quad\quad p_{0,j} := m_j. \end{cases}$$

Below is a pictorial representation of $p_{i,j}$.

Figure-1. *Pictogram of entry $p_{i,j}$ in ξ.*



Let $m_1 = 1$, $m_2 = 1$and $m_3 = 0$. All possible bases from this partition, as given by the construction of Figure 1, are

$$\begin{pmatrix}1 & & 1\\ & 1 & \end{pmatrix}, \quad \begin{pmatrix}1 & & 0\\ & 1 & \end{pmatrix}, \quad \begin{pmatrix}1 & & 0\\ & 0 & \end{pmatrix}.$$

Here, we discuss the module structure on K_n. Our representation space K_n is a sl_n −module. Although this is true, we will not prove it. It is a sl_n −module via actions of upper triangular matrices, lower triangular matrices and the diagonal matrices on ξ [1].

In Gelfand [1], a comprehensive construction was presented for the action of upper triangular, lower triangular and diagonal matrices on basis vector ξ. For upper triangular matrices in general, suppose $\xi^j_{k-1,k}$ is the pattern obtained from ξ by replacing $m_{i,k-1}$ with $m_{i,k-1}+1$, the upper triangular matrix $E_{k-1,k}$ acts on ξ as

$$E_{k-1,k}(\xi) = \sum_j a^j_{k-1,k}\,(\xi^j_{k-1,k}). \qquad (2)$$

For a 3×3matrix, the action of $E_{i,j}$ on ξ raises the i^{th} row in the basis ξ by 1 on every entry in that row accordingly. For the case $n = 3$, the formulas for computing the action can be found in Gelfand [1]. In general, $E_{i,j}$ is generated by $E_{i,i+1}$.

The action of $F_{i,j}$on ξ reduces the entries of the i^{th} row in ξ by 1 accordingly. This is done in such a way that rules governing the size of entries are observed. Suppose $\bar{\xi}^j_{k,k-1}$is the pattern obtained from ξ by replacing $m_{i,k-1}$ with $m_{i,k-1}-1$. The lower triangular matrix $F_{k,k-1}$acts on ξ as

$$F_{k,k-1}(\xi) = \sum_j b^j_{k,k-1}\,(\bar{\xi}^j_{k,k-1}). \qquad (3)$$

The formulas for $n = 3$ can be found in Gelfand [1]. In general, $F_{i+1,i}$ generates all $F_{i,j}$ and other actions can be computed using the Lie bracket operation.

The diagonal matrices can also be generated by $E_{i,i+1}$ and $F_{i+1,i}$. Some coefficients from the action of $H_{i,i}$ can be zero but not all coefficients. In general

$$H_{i,i}(\xi) = \left(\sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1}\right)(\xi) \text{ where } \left(\sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1}\right) \qquad (4)$$

is the coefficient of ξ. The formulas for computing the action of diagonal matrices $(H_{i,i})$ when $n = 3$ can be found in Gelfand [1].

Theorem 5 (sln_module). The representation space $K_n$is a sl_n −module.

A highest weight vector is the weight vector that is annihilated by every $(n\times n)$ upper triangular matrix (that is $E_{i,j}$ with $i<j$). We fixed ξ as our basis vector in K_n, the representation space where q is any integer depending on some conditions [1]. The nature of each basis vector depends on the dimension n of operator $E_{i,j}$ acting on it and the partition. For $n = 2$, we choose some integers $m_1, m_2 (m_1 \geq m_2)$ such that the condition $m_1 \geq q \geq m_2$ is satisfied. When $n = 3$, we choose three integers m_1, m_2, m_3 $(m_1 \geq m_2 \geq m_3)$. The bases vectors in the representation space are now numbered by triples, p_1, p_2, q. The representation is given by $m_1 \geq p_1 \geq m_2 \geq p_2 \geq m_3$ and $p_1 \geq q \geq p_2$. We have our bases vectors of the form

$$\xi = \begin{pmatrix}p_1 & & p_2\\ & q & \end{pmatrix}.$$

Every weight vector has a corresponding weight. The bases vectors are the weight vectors. Constructing these bases depends on the choices of $m_1 \geq m_2 \geq \cdots \geq m_n$ as defined above.

Suppose $H_{i,i}$ is a square diagonal matrix. The action $H_{i,i}(\xi) = \kappa_i(\xi)$, where $\kappa_i$is the eigenvalue of corresponding weight vector ξ. There is a map ω such that for $h \in H$, $\omega\colon h \to \mathbb{C}$ such that $H_{i,i} \mapsto \kappa$. The map ω is the weight. Now, for arbitrary partition $m_1 \geq m_2 \geq \cdots \geq m_n$, $H_{i,i}(\xi) = \kappa_i(\xi)$ has weight $\omega = \kappa_1\varepsilon_1 + \kappa_2\varepsilon_2 + \cdots + \kappa_n\varepsilon_n$ where $\varepsilon_1$is the weight for ξ_1, $\varepsilon_1 + \varepsilon_2$ the weight of ξ_2 and so on. Since sl_n is trace free, $\varepsilon_1 + \cdots + \varepsilon_n = 0$. In general,

$$H_{i,i}(\xi) = \left(\sum_{i=1}^{k} m_{i,k} - \sum_{i=1}^{k-1} m_{i,k-1}\right)(\xi).$$

Suppose we let $m_1 = 3$, $m_2 = 2$ and $m_3 = 0$. Then $E_{1,2}^{l_3}E_{2,3}^{l_2}E_{1,2}^{l_1}\xi_q = \beta$ where β is a highest weight vector and l_i is the maximum times each operator can act on ξ while all conditions are observed to either raise the first row or the second row of ξ. Due to the nature of transitions as a consequence of the action of the sequence, the result is unique (proved later).

The representation space K_n is simple if for all $v \in K_n$, there exist upper triangular square matrices such that $E^{\underline{a}(\alpha)} \cdot v = \beta$, a highest weight vector. The weight vectors could be of the form $v = \sum_{\alpha(\text{patterns})} a_\alpha \alpha$ where α is a weight vector. We will show that there exists a sequence of upper triangular matrices$E^{\underline{a}(\alpha)}$ such that its action on any sum of weight vectors annihilates all but one. That resulting weight vector is a highest weight vector.

2.1. Main Results

Theorem 6. The representation space $K_n$is a simple sl_n −module.

This theorem requires a proof for many parts so we break it down into two propositions and two lemmas.

Proposition 7. For every given partition there is a highest weight vector, β.

Proof. Suppose for integers $m_1, m_2, \ldots, m_n$ with $(m_1 \geq \cdots \geq m_n)$ that

$$\beta = \begin{pmatrix} m_1 & & m_2 & & m_3 & \cdots & m_n \\ & m_1 & & m_2 & & \cdots & \\ & & \ddots & & \ddots & & \\ & & & m_1 & & & \end{pmatrix}.$$

Suppose there exists some ξ_i such that (we have a total ordering)

$$\xi_1 < \xi_2 < \cdots < \xi_s$$

and $E_{i,i+1} \cdot \xi_1 \neq 0$ and $E_{i,i+1} \cdot \xi_2 \neq 0$ and so on. Also, suppose that entries in both basis vectors ξ_1, ξ_2 are equal at the bottom, except for a certain row such that in that row, the sum of the entries (for ξ_1, denote the first entry of the row $by\ {}^1\xi_1$ and the second entry $by\ {}^2\xi_1$ and so on)

$${}^1\xi_1 + {}^2\xi_1 + \cdots + \overset{s}{+}\xi_1 < {}^{a_1}\xi_2 + {}^{a_2}\xi_2 + \cdots + {}^{a_s}\xi_2.$$

and that

$${}^1\xi_1 < ({}^2\xi_1 + \cdots \overset{s}{+}\xi_1) < ({}^{a_1}\xi_2 \overset{a_2}{+} \xi_2 + \cdots \overset{a_s}{+} \xi_2).$$

We can write $\xi_1 = (v - \xi_i)$, where ξ_i is some weight vector and $v = \sum_{i=1}^{s} c_i\xi_i$, $c_i \neq 0$ is a complex number. The action

$$E_{i,i+1} \cdot v = E_{i,i+1} \cdot \xi_1 + E_{i,i+1} \cdot (v - \xi_1) = {}^1\xi_1 + \sum_{\Psi_{i,j} \overset{1}{>} \xi_1} \Psi_{i,j},$$

where $\Psi_{i,j}$ is the set of all resulting weight vectors the sum of whose entries in the nth row are greater than that of ${}^1\xi_1$. Now, with a sequence of upper triangular matrices which raises the entries of ξ_1,

$$E^{\underline{a}(\xi_1)} \cdot v = E^{\underline{a}(\xi_1)} \cdot \sum_i c_i\xi_i \text{ for } c_i \neq 0.$$

The sequence is actually raising the weight vectors by the series of actions and the supposedly the smallest basis vector becomes a highest weight vector as a consequence. So

$$E^{\underline{a}(\xi_1)} \cdot \sum_i c_i\xi_i = \lambda_\beta\beta + \sum_{\Psi_{i,j} > \beta} \Psi_{i,j} = \lambda_\beta\beta \text{ for } \lambda_\beta \neq 0.$$

Therefore, β is a highest weight vector.

The weight for $\beta$is such that

$$H_{i,i} \cdot \beta = \left(\sum_{k=1}^{i} m_{k,i} - \sum_{k=1}^{i-1} m_{k,i-1}\right) \cdot \beta$$

$$= [q + (p_1 + p_2 - q) + (m_1 + m_2 + m_3 - p_1 - p_2) + \cdots + q + (p_1 + p_2 - q) + \cdots + (m_1 + m_2 + \cdots + m_n - p_1 - p_2 - \cdots - p_{n-1})] \cdot \beta$$

$$= (c_1\varepsilon_1 + c_2\varepsilon_2 + c_3\varepsilon_3 + \cdots + c_{n-1}\varepsilon_{n-1} + c_n\varepsilon_n) \cdot \beta.$$

So β has weight $c_1\varepsilon_1 + c_2\varepsilon_2 + \cdots + c_{n-1}\varepsilon_{n-1} + c_n\varepsilon_n$, which is a highest weight. Q.E.D.

Proposition 8. For any basis vector ξ, there exists a set of upper triangular matrices, $E^{\underline{a}(\xi)}$, such that

$$E^{\underline{a}(\xi)}.\xi = \lambda_\beta\beta \text{ for } \lambda_\beta \neq 0,$$

where

$$E^{\underline{a}(\xi)} = E_{1,2}^{a_N}(E_{2,3}^{a_{N-1}}E_{1,2}^{a_{N-2}}) \cdots (E_{n-1,n}^{a_{n-1}} \cdots E_{2,3}^{a_2}E_{1,2}^{a_1})$$

Proof. From the order $\xi_1 < \xi_2 < \cdots < \xi_s$introduced in Proposition 7, we see that ξ_1 is smaller than all other basis vectors. The action

$$E^{\underline{a}(\xi)}.\xi = E^{\underline{a}(\xi_1)} \cdot \xi_1 + E^{\underline{a}(\xi_1)} \cdot \left(\sum_{\Psi_{i,j} > \xi_1} \Psi_{i,j} \right).$$

But $(\sum_{\Psi_{i,j}>\xi_1} \Psi_{i,j})$ will be annihilated by the action since its elements are bigger and $E^{\underline{a}(\xi)}$ will be the sequence that raises ξ_i to β, which is a highest weight. Q.E.D.

Lemma 9. Suppose v is a non-zero element in K_n,

$$v = \sum_{i=1}^{s} c_i \xi_i, \text{with } c_i \neq 0 \in \mathbb{C}.$$

Then there exists a sequence of upper triangular matrices such that

$$E^{\underline{a}(v)} \cdot v = \lambda_\beta \beta \text{ where } \lambda_\beta \neq 0.$$

Proof. From Proposition 7, for $\xi_1 < \xi_2$, we established that

$${}^1\xi_1 < ({}^2\xi_1 + \cdots \overset{s}{+} \xi_1) < ({}^{a_1}\xi_2 \overset{a_2}{+} \xi_2 + \cdots \overset{a_s}{+} \xi_2).$$

Then, for all $c_i \neq 0$,

$$E^{\underline{a}(\xi_i)} \cdot v \;\; = E^{\underline{a}(\xi_i)} \cdot \left(\sum_{i=1}^{s} c_i \xi_i \right) = E^{\underline{a}(\xi_i)} \cdot \left(c_1 \, {}^1\xi_1 + \sum_{\xi_i > {}^1\xi_1} c_i \xi_i \right).$$

Since ${}^1\xi_1$is the smallest basis, the action will be

$$E^{\underline{a}(\xi_i)} \cdot v \;\; = \lambda_\beta \beta + \sum_{\Gamma_{i,j} > \beta} \lambda_i \Gamma_{i,j} = \lambda_\beta \beta.$$

Therefore, $E^{\underline{a}(\xi_i)} \cdot v = \lambda_\beta \beta$. Q.E.D.

This implies

Corollary 10. If $S \subset K_n$is a non-zero submodule, then $\beta \in S$.

We proved from Proposition 7 that there is a highest weight vector $\beta \in K_n$. So if $S \subset K_n$is a non-zero submodule, then $\beta \in S$.

Let M be a simple finite-dimensional module and v be a highest weight vector, the following result claims that M is generated by v through applying iterative lower triangular matrices on v. We can view this iterated applying as being a product in some algebra (namely the universal enveloping algebra).

Definition 11 (Monomial Basis). For M a finite-dimensional module and v a highest weight vector, consider the fixed basis $F_{i,j}$ and the monomials in these $F_{i,j}$ only. A given set B of monomials is called a monomial basis of M if $\{F^{\underline{b}}.v \mid F^{\underline{b}} \in B\}$is a basis of M, where $F^{\underline{b}}$ is a product (sequence) of some lower triangular matrices.

Lemma 12. Let

$$B = \{F^{\underline{a}(\xi)} \mid \xi \text{ is a basis element}\}.$$

Then

$$\mathcal{B} = \{F^{\underline{a}(\beta)} \cdot \beta \mid F^{\underline{a}(\beta)} \in B\}$$

is a basis.

Proof. Now, we want to show that K_n is generated by β. From the ordering in Proposition 7, we see that at least (for the lower triangular matrix $F_{i+1,i}$)

$$F_{i+1,i} \overset{1}{\cdot} \xi_1 < F_{i+1,i} \cdot \xi_2.$$

Also, for $j \neq i$

$$F_{j+1,j} \overset{1}{\cdot} \xi_1 < F_{j+1,j} \cdot \xi_2.$$

Therefore, we can write

$$F^{\underline{a}(\beta)} \cdot \beta = \chi_i + \sum_{\gamma_{i,j} > \chi_i} \gamma_{i,j}$$

where $\xi_1 = \chi_i$ and $\xi_2 = \sum_{\gamma_{i,j}>\chi_i} \gamma_{i,j}$.

Suppose

$$\sum_{i=1}^{N} c_i F^{\underline{a}(\beta)} \cdot \beta = 0,$$

where $N$is the size of the basis ξ_i, and all $c_i \neq 0$. Then

$$\sum_{i=1}^{N} c_i \left(\chi_i + \sum_{\gamma_{i,j} > \chi_i} \gamma_{i,j} \right) = 0.$$

We can fix χ_i such that $\chi_1 < \chi_2 < \cdots < \chi_N$. So

$$\sum_{i=1}^{N} c_i \left(\chi_i + \sum_{\gamma_{i,j} > \chi_i} \gamma_{i,j} \right) = c_1 \chi_1 + \sum_{i=2}^{N} c_i \left(\chi_i + \sum_{\gamma_{i,j} > \chi_i} \gamma_{i,j} \right) + \gamma_1 = 0.$$

We know $\chi_1$is the smallest and $\{\chi_i\}$ are linearly independent for $1 \leq i \leq N$, then $c_1 = 0$. Therefore, the set

$$\{F^{\underline{a}(\beta)} \cdot \beta \mid F^{\underline{a}(\beta)} \in B\}$$

is linearly independent.

We are given that ξ_i is a basis of K_n implying ξ (β in particular) is a basis element. The cardinality of ξ is D (that is dim K_n, in other words the number of basis vectors one can make from a given partition). Since ξ has N linearly independent elements, then dim $\langle F_{\cdot}^{a(\beta)} \cdot \beta \rangle = dim\, K_n = D$. So $\{F^{\underline{a}(\beta)} \cdot \beta \mid F^{\underline{a}(\beta)} \in B\}$ spans and is a basis in K_n. Since $\{F^{\underline{a}(\beta)} \cdot \beta \mid F^{\underline{a}(\beta)} \in B\}$ spans K_n and all its elements are linearly independent, then it is all of K_n. Therefore, the weight vector β generates all of K_n. Q.E.D.

From the above proofs, we can make out that if β is a highest weight vector, S a submodule of K_n (i.e $\beta \in S$ and S is all of K_n) implies β generates all of K_n. Therefore, there is no invariant subspace of K_n.

Corollary 13. The representation space K_n is generated by β, and moreover if $S \subset K_n$ is a non-zero submodule, then $S = K_n$.

This completes the proof for Theorem 6. So, the representation space K_n is a simple sl_n −module. Already, a monomial basis is constructed in Lemma 12.

3. Conclusion

In this paper, our representation is actually $\rho: sl_n \to \mathrm{End}(K_n)$ where $x \mapsto \rho(x)$. The map ρ is linear and also the identity. Suppose $v \in K_n$ and $v = \lambda_1\xi_1 + \cdots + \lambda_n\xi_n$, where $\xi_1, \cdots, \xi_n$ are basis vectors and $\lambda_1, \cdots, \lambda_n$ are non-zero coefficients. Let $\lambda_1 \neq 0$ and $\lambda_2 = \cdots = \lambda_n$. Then $F_{i,j} \cdot v = \lambda_1 F_{i,j} \cdot \xi_1$ and $E_{i,j} \cdot (F_{i,j} \cdot v) = \lambda_1 E_{i,j} \cdot (F_{i,j} \cdot \xi_1)$ are both well defined operations in our representation. Now, let $\lambda_1 = \lambda_3 = \cdots = \lambda_n$ and $\lambda_2 \neq 0$. Then $E_{i,j} \cdot v = \lambda_2 E_{i,j} \cdot \xi_2$ and $F_{i,j} \cdot (E_{i,j} \cdot v) = \lambda_2 F_{i,j} \cdot (E_{i,j} \cdot \xi_2)$ again are both well defined operations in our representation. The diagonal matrices act by a scalar; that is $H_{i,i} \cdot \xi = \kappa\xi$. In all the actions above, the results are all accounted for in formulas of Equations (2), (3) and (4). If $\xi_1, \cdots, \xi_n \in S$, then S is all of K_n. So, ρ has no invariant subspace. Therefore, ρ is an irreducible representation of the special linear algebra, sl_n.

For any partition, we can construct all possible basis vectors and modules as discussed above. We apply total ordering on basis vectors to identify the smallest basis vector. A sequence of upper triangular matrices that acts maximally on the smallest bases vector will eventually act on a set of bases vectors resulting in a total annihilation of all bases vectors but raising the smallest basis vector maximally, to a highest weight vector which has weight $\omega_i = c_1\varepsilon_1 + \cdots + c_n\varepsilon_n$. We also proved that every basis vector has a sequence of upper triangular matrices that acts on it maximally to yield a highest weight vector. We proved the existence of monomial basis and gave a construction. Each of these results contributes in proving our main result, that sl_n −module is simple, and has monomial basis.

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References

[1] Gelfand, Z. M. L., 2020. *Finite-dimensional representations of the group of unimodular matrices*. Semantic Scholar.

[2] Futorny, V., 2015. "Irreducible Generic Gelfand-Tsetlin Modules of gl(n)." *SIGMA*, Available: http://doi:10.3842/SIGMA.2015.018

[3] Futorny, V., Grantcharov, D., and Ramirez, L. E., 2016. "Singular Gelfand-Tsetlin modules of gl(n)." *Advances in Mathematics*, vol. 290, pp. 453-482. Available: http://doi:10.1016/j.aim.2015.12.001

[4] Carter, R. and Carter, R. W., 2005. *Lie algebras of finite and affine type*. Cambridge University Press.